School of Economics, Management and Statistics

University of Bologna

Academic Year 2017/18

NOTES on LINEAR ALGEBRA[1]

for the students of Stats and Maths

[1]This is a modified version of the notes by Prof Laura Guidotti, translated by Alessandra Giovagnoli.

Contents

Chapter 1

1 The vector space $\mathbb{R}^n$

1.1 Definition of vector space

A **vector space V over the real numbers** $\mathbb{R}$ is a collection of objects called **vectors**, which may be added together and multiplied by real numbers, called **scalars**. The operations of vector addition and scalar multiplication must satisfy the following requirements (*axioms*). Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be arbitrary vectors in $\mathbf{V}$, and α and β arbitrary scalars in $\mathbb{R}$

- *Commutativity of addition*

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

- *Associativity of addition*

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

- *Identity element of addition* There exists an element $\mathbf{0} \in \mathbf{V}$, called the **zero vector**, such that

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

  for all $\mathbf{u} \in \mathbf{V}$.

- *Inverse elements of addition* For every $\mathbf{u} \in \mathbf{V}$, there exists an element $-\mathbf{u} \in \mathbf{V}$, called the **opposite vector** of $\mathbf{u}$, such that

$$\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}.$$

- *Distributivity of scalar multiplication with respect to vector addition*

$$\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$$

  for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ and for all $\alpha \in \mathbb{R}$

- *Distributivity of scalar multiplication with respect to addition of real numbers*

$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$$

  for all $\mathbf{v} \in \mathbf{V}$ and for all $\alpha, \beta \in \mathbb{R}$

- *Compatibility of scalar multiplication with real multiplication*

$$\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$$

  for all $\mathbf{v} \in \mathbf{V}$ and for all $\alpha, \beta \in \mathbb{R}$

- *Identity element of scalar multiplication*

$$1\mathbf{v} = \mathbf{v},$$

for all $\mathbf{v} \in \mathbf{V}$, where $1 \in \mathbb{R}$ is the multiplicative identity.

Straightforward consequences of the axioms are:

- $\alpha\mathbf{0} = \mathbf{0}$
- $0\mathbf{v} = \mathbf{0}$
- $(-\alpha)\mathbf{v} = \alpha(-\mathbf{v}) = -(\alpha\mathbf{v})$
- $\alpha\mathbf{v} = \mathbf{0} \Rightarrow$ either $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$.

The mathematical definition of vector space.is very general and includes a wide variety of mathematical objects (e.g. the set of all the real functions is a vector space over the reals) but we are going to deal just with one important type, namely the *numerical* vector spaces $\mathbb{R}^n$, for all $n \geq 1$, and their subspaces

1.2 The numerical space $\mathbb{R}^n$

Given the set of real numbers $\mathbb{R}$ and a natural number n, $n \geq 1$, we write:

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n), x_i \in \mathbb{R}, i = 1, \cdots, n\}$$

for the set of all ordered $n-$tuples of real numbers. The elements of $\mathbb{R}^n$ are called *points*. The origin O is the point $(0, 0, ..., 0)$.

The space $\mathbb{R}^n$ is subject to intuitive physical interpretations when $n = 1, 2, 3$. In detail:

1. $\mathbb{R}^1$ (i.e. $\mathbb{R}$) can be thought of as an algebraic model for the set of points of a (straight) line;
2. $\mathbb{R}^2$ can be thought of as an algebraic model for the set of points of a plane;
3. $\mathbb{R}^3$ can be thought of as an algebraic model for the set of points of the physical 3-dimensional space where we live.

These identifications depend on the choice of a (Cartesian) *coordinate system*, as we will see in the following. Geometric objects of the real line, plane or space are described by algebraic expressions and operations in $\mathbb{R}^1, \mathbb{R}^2$and $\mathbb{R}^3$ respectively.

For $n \geq 4$ there is no physical-geometrical interpretation, but $\mathbb{R}^n$ is a useful mathematical model to study a variety of different real life phenomena that depend on a large number of variables.

Now we come to *vectors.* In $\mathbb{R}^3$ (also $\mathbb{R}^1, \mathbb{R}^2$) a vector can be thought of as an **action** that shifts points in space. All the points are shifted in the same direction and verse and by the same distance. **We can identify each vector with the point where the origin O is shifted to.**

Example 1.1. 1. Given a shift of all the points of $\mathbb{R}^2$ such that the origin $(0,0)$ is shifted to point $(1,3)$, we describe this by a vector $\mathbf{v} = (1,3)$. The same vector $\mathbf{v} = (1,3)$ will shift point $(-1,2)$ to $(0,5)$.

2. In $\mathbb{R}^3$, $(1,-2,4)$ is mapped to $(6,5,3)$ by the vector $(5,7,-1)$.

We need to extend this idea to $\mathbb{R}^n$ $\forall n$. In $\mathbb{R}^n$ if O is mapped to $(x_1, x_2, \ldots, x_n)$ by the vector $\mathbf{v}$, we write

$$\mathbf{v} = (x_1, x_2, \ldots, x_n).$$

Vectors are usually indicated by letters in bold, e.g. $\mathbf{v}$, or underlined, namely $\underline{\mathbf{v}}$, or sometimes with a little arrow over them, like this: $\overrightarrow{v}$. The numbers $x_1, x_2, \ldots, x_n$ are called the *components* or *coordinates* of the vector. Point $(x_1, x_2, \ldots, x_n)$ is called the *position point* of $\mathbf{v}$. If a vector is identified by the action of shifting point A to point B, (respectively, the *initial* and *end point* of the vector) we write $\overrightarrow{AB}$. If point $(x_1, x_2, \ldots, x_n)$ is shifted to point $(y_1, y_2, \ldots, y_n)$ by a vector $\mathbf{u}$, then the components of $\mathbf{u}$ are $(y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n)$.

Two vectors $\mathbf{u} = (x_1, x_2, \ldots, x_n)$, $\mathbf{v} = (y_1, y_2, \ldots, y_n)$ are equal if and only if $x_i = y_i, \quad i = 1, \ldots, n$.

Now we can say that $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n), x_i \in \mathbb{R}, i = 1, \cdots, n\}$ is also a **set of vectors**. It may be confusing to have the same notation for points and vectors. Later, we shall see that it is more useful to write vectors as vertical n-tuples:

$$\begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix},$$

but in this chapter we stick to the horizontal notation.

Particularly

1. $\mathbb{R}$ can be thought of as an algebraic model also for the set of geometric vectors belonging to the same line and having the same initial point;

2. $\mathbb{R}^2$ can be thought of as an algebraic model also for the set of geometric vectors whose directions lie on a plane and have the same initial point (of the plane);

3. $\mathbb{R}^3$ can be thought of as an algebraic model also for the set of all geometric vectors of the physical 3-dimensional space having the same initial point.

In order to be able to say that $\mathbb{R}^n$ is a **vector space** over $\mathbb{R}$ operations must be defined between vectors and between vectors and scalars of $\mathbb{R}$, namely addition of two vectors and multiplication of a vector by a scalar, and they must satisfy the axioms of Section 1.1.

1.2.1 Addition

Take any two vectors $\mathbf{u} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{v} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, which we can think of as acting on the points, and imagine performing both actions (shifts) one after the other. For instance given $\mathbf{u} = (3, 2)$ and $\mathbf{v} = (1, -4)$ in $\mathbb{R}^2$, first the origin is mapped to $(3, 2)$ by $\mathbf{u}$, and then point $(3, 2)$ is mapped to point $(4, -2)$ by $\mathbf{v}$. As a combined result we get that the origin is mapped to point $(4, -2)$ by a new vector $\mathbf{w}$.

We call this combination of vectors their *addition*. The vector $\mathbf{w}$ that performs the combined shift is called the *sum of* $\mathbf{u}$ and $\mathbf{v}$ and is denoted by $\mathbf{u} + \mathbf{v}$.

Definition 1.1.

$$\mathbf{u} + \mathbf{v} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Namely, the sum of two vectors is the vector whose components are the sums of the components with the same subscripts.

In the example above, $\mathbf{w} = (3 + 1, 2 - 4) = (4, -2)$.

Properties of the addition in $\mathbb{R}^n$:

- *Commutative property.* This property is satisfied because

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ &= (y_1 + x_1, y_2 + x_2, \ldots, y_n + x_n) \\ &= \mathbf{v} + \mathbf{u}. \end{aligned}$$

  for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

- *Associative property* This property is satisfied because

$$\begin{aligned} (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= ((x_1 + y_1) + w_1, (x_2 + y_2) + w_2, \ldots, (x_n + y_n) + w_n) \\ &= (x_1 + (y_1 + w_1), x_2 + (y_2 + w_2), \ldots, x_n + (y_n + w_n)) \\ &= \mathbf{u} + (\mathbf{v} + \mathbf{w}) \end{aligned}$$

  for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$

- *The zero vector property* This property is satisfied because $\mathbf{0} = (0, 0, \cdots, 0)$ is such that

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

  for all $\mathbf{u} \in \mathbb{R}^n$.

- *The opposite vector property.* This property is satisfied because for any $\mathbf{u} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ the vector $-\mathbf{u} = (-x_1, -x_2, \ldots, -x_n)$ is such that
$$\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}.$$

1.2.2 Scalar multiplication

Definition 1.2. For all vectors $\mathbf{v} \in \mathbb{R}^n$ and real numbers $\alpha \in \mathbb{R}$, define
$$\alpha\mathbf{v} = \alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n).$$

Properties

- *Distributivity of scalar multiplication with respect to vector addition.* This property is satisfied because
$$\begin{aligned} \alpha(\mathbf{u} + \mathbf{v}) &= \alpha(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ &= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2, \ldots, \alpha x_n + \alpha y_n) \\ &= \alpha\mathbf{u} + \alpha\mathbf{v} \end{aligned}$$

- *Distributivity of scalar multiplication with respect to addition of real numbers.* This property is satisfied because
$$\begin{aligned} (\alpha + \beta)\mathbf{v} &= ((\alpha + \beta)y_1, (\alpha + \beta)y_2, \ldots, (\alpha + \beta)y_n) \\ &= \alpha(y_1, y_2, \ldots, y_n) + \beta(y_1, y_2, \ldots, y_n) \\ &= \alpha\mathbf{v} + \beta\mathbf{v} \end{aligned}$$

- *Compatibility of scalar multiplication with real multiplication* This property is satisfied because
$$\begin{aligned} \alpha(\beta\mathbf{v}) &= \alpha(\beta y_1, \beta y_2, \ldots, \beta y_n) \\ &= (\alpha\beta y_1, \alpha\beta y_2, \ldots, \alpha\beta y_n) \\ &= (\alpha\beta)\mathbf{v} \end{aligned}$$

- *Identity element of scalar multiplication.* This property is satisfied because
$$1\mathbf{v} = (1y_1, 1y_2, \ldots, 1y_n) = \mathbf{v}$$
where $1 \in \mathbb{R}$ is the multiplicative identity. For all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and for all $\alpha, \beta \in \mathbb{R}$

Because of the above operations (addition and multiplication by a scalar) and their properties, we can say that $\mathbb{R}^n$ is a **vector space** over $\mathbb{R}$. Let us look at some of its main features.

1.2.3 Linear combinations

We are going to combine the above operations into one.

Definition 1.3. Given the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$, and the scalars $\alpha_1, \alpha_2, \cdots, \alpha_r$, combining addition and scalar multiplication we get the vector:

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r$$

which is called a *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ with *coefficients* (or *weights*) $\alpha_1, \alpha_2, \cdots, \alpha_r$.

Example 1.2.

1. In $\mathbb{R}^4$ the linear combination of vectors $\mathbf{v}_1 = (1, 0, -2, 0), \mathbf{v}_2 = (3, 4, -1, 2)$, $\mathbf{v}_3 = (0, 2, 0, 1)$ with coefficients $\lambda_1 = -1, \lambda_2 = 6, \lambda_3 = 2$ is:

$$-1\mathbf{v}_1 + 6\mathbf{v}_2 + 2\mathbf{v}_3 = (17, 28, -4, 14).$$

2. In $\mathbb{R}^2$, the linear combination of vectors $\mathbf{v} = (1, 3)$ and $\mathbf{w} = (3, 9)$ with weights $\lambda = -3, \mu = 1$ is:

$$-3\mathbf{v} + \mathbf{w} = (-3 + 3, -9 + 9) = (0, 0) = \mathbf{0}.$$

3. In $\mathbb{R}^3$, we can ask ourselves whether vector $\mathbf{u} = (2, -1, 0)$ is or is not a linear combination of the vectors $\mathbf{v} = (1, -3, 0)$ and $\mathbf{w} = (0, 1, 0)$, i.e. whether there exist two scalars α and β such that:

$$\mathbf{u} = \alpha\mathbf{v} + \beta\mathbf{w}$$

namely

$$(2, -1, 0) = \alpha(1, -3, 0) + \beta(0, 1, 0) = (\alpha, -3\alpha, 0) + (0, \beta, 0) = (\alpha, -3\alpha + \beta, 0)$$

hence

$$\alpha = 2, \quad \beta = 5.$$

Thus vector $\mathbf{u}$ is indeed a linear combination of $\mathbf{v}$ and $\mathbf{w}$: $\mathbf{u} = 2\mathbf{v} + 5\mathbf{w}$. It is of interest to point out that $\mathbf{v}$ too is a linear combination of $\mathbf{u}$ and $\mathbf{w}$, since from $\mathbf{u} = 2\mathbf{v} + 5\mathbf{w}$, one obtains

$$\mathbf{v} = \frac{1}{2}\mathbf{u} - \frac{5}{2}\mathbf{w},$$

and also that $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{w} = \frac{1}{5}\mathbf{u} - \frac{2}{5}\mathbf{v}.$$

4. Every vector of $\mathbb{R}^3$ is a linear combination of the three vectors $\mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0), \mathbf{e}_3 = (0, 0, 1)$, since

$$\begin{aligned} \mathbf{v} &= (x_1, x_2, x_3) = (x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3) = \\ &= x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1) = \\ &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3. \end{aligned}$$

More in general, every vector of $\mathbb{R}^n$ is a linear combination of the n vectors:

$$\mathbf{e}_1 = (1, 0, \cdots, 0), \mathbf{e}_2 = (0, 1, \cdots, 0), \cdots, \mathbf{e}_n = (0, 0, \cdots, 1)$$

as follows:

$$\mathbf{v} = (x_1, x_2, \ldots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

Given a set of vectors in $\mathbb{R}^n$, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, the linear combination with all the coefficients equal to zero is the zero (or null) vector $\mathbf{0}$:

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_r = \mathbf{0}.$$

This is called the *trivial* linear combination.

1.2.4 Linear dependence

Definition 1.4. Given r vectors of $\mathbb{R}^n$

$$\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r$$

the vectors of the set $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r\}$ are said to be *linearly dependent* if there are scalars $\lambda_1, \lambda_2, \cdots, \lambda_r$, **not all zero** such that:

$$\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \cdots + \lambda_r\mathbf{a}_r = \mathbf{0}$$

i.e. there exists a non-trivial linear combination which gives the zero vector. A set of vectors are *linearly independent* when they are not linearly dependent, i.e. the **only** linear combination of all the vectors of the set giving the zero vector is the trivial one.

Example 1.3.

1. In $\mathbb{R}^3$ the vectors

   $$\mathbf{a}_1 = (1, -1, 0), \mathbf{a}_2 = (2, 0, 1), \mathbf{a}_3 = (-1, -3, -2)$$

   are linearly dependent since

   $$3\mathbf{a}_1 - 2\mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}.$$

2. In $\mathbb{R}^4$ the vectors of the set :

   $$B = \{\mathbf{v}_1 = (1, 0, 2, 1), \mathbf{v}_2 = (0, 1, 1, 0), \mathbf{v}_3 = (1, 1, 1, 1)\}$$

   are linearly independent. To show this, write

   $$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 = (\lambda_1 + \lambda_3, \lambda_2 + \lambda_3, 2\lambda_1 + \lambda_2 + \lambda_3, \lambda_1 + \lambda_3) = (0, 0, 0, 0)$$

We need to solve a set of linear equations in the unknowns $\lambda_1, \lambda_2, \lambda_3$

$$\begin{cases} \lambda_1 + \lambda_3 = 0 \\ \lambda_2 + \lambda_3 = 0 \\ 2\lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_1 + \lambda_3 = 0 \end{cases}$$

equivalent to:

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{cases}$$

So the only linear combination is the one with coefficients equal to zero.

3. In $\mathbb{R}^n$ the set of vectors $\{\mathbf{e}_1 = (1, 0, \cdots, 0), \mathbf{e}_2 = (0, 1, \cdots, 0), \cdots, \mathbf{e}_n = (0, 0, \cdots, 1)\}$ is linearly independent:

   $\lambda_1\mathbf{e}_1+\lambda_2\mathbf{e}_2+\cdots+\lambda_n\mathbf{e}_n = (\lambda_1, \lambda_2, \cdots, \lambda_n) = \mathbf{0} \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$

4. If a set contains just one vector $\mathbf{v}$, from:

   $$\alpha\mathbf{v} = \mathbf{0} \Rightarrow \alpha = 0 \text{ or } \mathbf{v} = \mathbf{0}$$

   there follows that the set $\{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} = \mathbf{0}$.

Proposition 1.1. *If a set $A = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r\}$ contains at least two vectors, the vectors of A are linearly dependent if and only if at least one of them is a linear combination of the others.*

Proof If

$$\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \cdots + \lambda_r\mathbf{a}_r = \mathbf{0}$$

and, say, $\lambda_r \neq 0$, then vector $\mathbf{a}_r$ is a linear combination of the first $r-1$ vectors of A, since

$$\mathbf{a}_r = -\frac{\lambda_1}{\lambda_r}\mathbf{a}_1 - \frac{\lambda_2}{\lambda_r}\mathbf{a}_2 + \cdots - \frac{\lambda_{r-1}}{\lambda_r}\mathbf{a}_{r-1}.$$

Conversely if

$$\mathbf{a}_r = \mu_1\mathbf{a}_1 + \mu_2\mathbf{a}_2 + \cdots + \mu_{r-1}\mathbf{a}_{r-1},$$

the equality

$$\mu_1\mathbf{a}_1 + \mu_2\mathbf{a}_2 + \cdots + \mu_{r-1}\mathbf{a}_{r-1} - \mathbf{a}_r = \mathbf{0}$$

shows a non-trivial linear combination (the coefficient of vector $\mathbf{a}_r$ is $-1 \neq 0$) of the vectors of A. ■

In particular two vectors $\mathbf{v}, \mathbf{w}$ are linearly dependent if and only if $\mathbf{v} = \alpha\mathbf{w}$; in short, if the two vectors are *proportional* (also called *parallel*).

The following is an important result

Proposition 1.2. *Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r\}$ and $A \neq \{\mathbf{0}\}$, then it is possible to find a* ***maximal independent subset*** *$B \subseteq A$ namely a subset B such that*

- *the vectors of B are linearly independent;*
- *if $\mathbf{a}_i \in A$ and $\mathbf{a}_i \notin B$, the vectors of $B \cup \{\mathbf{a}_i\}$ are linearly dependent.*

Example 1.4. 1. In $\mathbb{R}^4$ we look for a maximal independent subset of

$$A = \{\mathbf{a}_1 = (1,0,1,0),\ \mathbf{a}_2 = (0,-1,2,1), \mathbf{a}_3 = (0,0,0,0),\\ \mathbf{a}_4 = (1,-1,3,1),\ \mathbf{a}_5 = (1,0,0,0)\};$$

A is a dependent set, since it contains the zero vector. The vector $\mathbf{a}_1$ is linearly independent because it is non-zero; the vectors $\{\mathbf{a}_1, \mathbf{a}_2\}$ are independent, because they are not proportional; the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ are linearly dependent. Also $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ are linearly dependent because $\mathbf{a}_4 = (1,-1,3,1) = \mathbf{a}_1 + \mathbf{a}_2$; lastly, $B = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ is a set of linearly independent vectors. Starting from a vector different from $\mathbf{a}_1$, one obtains a maximal independent set different from B, for instance $C = \{\mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_5\}$.

2. What about $E = \{\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1)\}$ in $\mathbb{R}^3$? Are they independent? What is a maximal independent subset?

1.3 Subspaces of $\mathbb{R}^n$

In the vector space $\mathbb{R}^n$ we are going to consider other vector spaces that are contained in it.

Definition 1.5. We say that a non empty subset $\mathbf{S} \subseteq \mathbb{R}^n$is a **vector subspace** if the following holds:

- for all $\mathbf{v_1}, \mathbf{v_2} \in \mathbf{S} \Rightarrow \mathbf{v}_1 + \mathbf{v}_2 \in \mathbf{S}$,
- for all $\alpha \in \mathbb{R}$, and for all $\mathbf{v} \in \mathbf{S} \Rightarrow \alpha\mathbf{v} \in \mathbf{S}$.

In other words, a subset of vector space $\mathbb{R}^n$ is a subspace if it contains all the linear combinations of its vectors. It can be shown that all the properties of a vector space still hold in a vector subspace too.

Special subspaces of $\mathbb{R}^n$ are
i) $\{\mathbf{0}\}$, the null subspace, containing only the zero vector;
ii) $\mathbb{R}^n$ itself;
iii) the subspace of all the vectors proportional to a given non-null one: $\{\lambda\mathbf{v}_0, \lambda \in \mathbb{R}\}$. In the physical 3−dimensional space this subspace corresponds to a line through the origin in the direction of $\mathbf{v}_0$;

iv) subsets consisting of all the vectors with certain coordinates equal to zero, e.g. those for which the second coordinate is 0, like:

$$\{\mathbf{v} = (x, 0, z); \quad x, z \in \mathbb{R}\}.$$

In the physical 3−dimensional space with Cartesian coordinates this subspace corresponds to the xz-plane.

Example 1.5. 1. Consider the following subsets of $\mathbb{R}^2$:

$$\begin{aligned} \mathbf{S} &= \{\mathbf{v} = (x, 2x); \quad x \in \mathbb{R}\} \\ \mathbf{T} &= \{\mathbf{v} = (x, 1); \quad x \in \mathbb{R}\} \\ \mathbf{W} &= \{\mathbf{v} = (0, y); \quad y \in \mathbb{R}\}. \end{aligned}$$

- $\mathbf{S}$ is a subspace (see iii) above), since given any two vectors of $\mathbf{S}$, $(x, 2x)$ and $(x', 2x')$, their sum $(x, 2x) + (x', 2x') = (x + x', 2(x + x'))$ belongs to $\mathbf{S}$, and so does the product $\alpha(x, 2x) = (\alpha x, 2\alpha x)$.
- The set $\mathbf{T}$ is not a subspace of $\mathbb{R}^2$, since $(x, 1) + (x', 1) = (x + x', 2)$ is not a vector of $\mathbf{T}$.
- The set $\mathbf{W}$ is a subspace of $\mathbb{R}^2$ (see iv) above).

2. Consider the following subsets of $\mathbb{R}^4$:

$$\begin{aligned} \mathbf{S} &= \{\mathbf{u} = (x, 2x, 0, 0); \quad x \in \mathbb{R}\} \\ \mathbf{T} &= \{\mathbf{v} = (\alpha, \beta, \alpha + \beta, 2\beta); \quad \alpha, \beta \in \mathbb{R}\} \\ \mathbf{W} &= \{\mathbf{w} = (\alpha, \alpha + 3, \alpha, 0); \quad \alpha \in \mathbb{R}\}. \end{aligned}$$

$\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbb{R}^4$, $\mathbf{W}$ is not, since it is not *closed* with respect to either addition or scalar multiplication, meaning that the sum of two vectors of $\mathbf{W}$ is not necessarily a vector of $\mathbf{W}$, nor is the product of a vector of $\mathbf{W}$ by a scalar.

Remark If S is a subspace of $\mathbb{R}^n$ then it contains the zero vector: to see this take a vector $\mathbf{v} \in S$ and $0 \in \mathbb{R}$, then $0\mathbf{v} = \mathbf{0} \in S$. The converse statement is not true: for example the set $W = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\} \subset \mathbb{R}^2$ contains $(0, 0)$, but it is not a subspace. In fact $(1, 1), (2, 4) \in W$ but $(1, 1) + (2, 4) = (3, 5) \notin W$.

Definition 1.6. The *intersection* $\mathbf{S}_1 \cap \mathbf{S}_2$ of two vector subspaces $\mathbf{S}_1$ and $\mathbf{S}_2$ of $\mathbb{R}^n$ is the set of all the vectors that belong to both:

$$\mathbf{S}_1 \cap \mathbf{S}_2 = \{\mathbf{u} \text{ such that } \mathbf{u} \in \mathbf{S}_1, \mathbf{u} \in \mathbf{S}_2\}.$$

It is never empty, since it contains at least the zero vector. It is straightforward to check that:

Proposition 1.3. *For any two vector subspaces* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *of* $\mathbb{R}^n$, $\mathbf{S}_1 \cap \mathbf{S}_2$ *is a vector subspace of* $\mathbb{R}^n$.

On the contrary, the *union* of two vector subspaces, $\mathbf{S}_1 \cup \mathbf{S}_2$, is **not** in general a subspace. However, we can talk about the *sum* of two subspaces.

Definition 1.7. For any two vector subspaces $\mathbf{S}_1$ and $\mathbf{S}_2$ of $\mathbb{R}^n$, define their *sum* $\mathbf{S}_1 + \mathbf{S}_2$ to be the set of all the sums $\mathbf{u} + \mathbf{v}$ of a vector $\mathbf{u} \in \mathbf{S}_1$ and a vector $\mathbf{v} \in \mathbf{S}_2$:

$$\mathbf{S}_1 + \mathbf{S}_2 = \{\mathbf{u} + \mathbf{v}, \text{ for all } \mathbf{u} \in \mathbf{S}_1, \mathbf{v} \in \mathbf{S}_2\}.$$

When $\mathbf{S}_1 \cap \mathbf{S}_2 = \{\mathbf{0}\}$ the sum of two subspaces is called a *direct sum*, denoted as $\mathbf{S}_1 \oplus \mathbf{S}_2$.

It is also straightforward to check that:

Proposition 1.4. *For any two vector subspaces* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *of* $\mathbb{R}^n$, $\mathbf{S}_1 + \mathbf{S}_2$ *is a vector subspace of* $\mathbb{R}^n$.

Example 1.6.

1. Consider the following subspaces of $\mathbb{R}^4$:

$$\begin{aligned} \mathbf{S} &= \{\mathbf{u} = (\gamma, 2\gamma, 0, 0); \quad \gamma \in \mathbb{R}\}, \\ \mathbf{T} &= \{\mathbf{v} = (0, \alpha, \beta, \alpha + \beta); \quad \alpha, \beta \in \mathbb{R}\} \end{aligned}$$

   Their intersection $\mathbf{S} \cap \mathbf{T}$ is $\{\mathbf{0}\}$ and the sum is

$$\mathbf{S} \oplus \mathbf{T} = \{(\gamma, \alpha + 2\gamma, \beta, \alpha + \beta) \quad \alpha, \beta, \gamma \in \mathbb{R}\}.$$

   This is not the whole space $\mathbb{R}^4$, since for instance $(0, -1, 0, 1) \in \mathbb{R}^4$ but $(0, -1, 0, 1) \notin \mathbf{S} \oplus \mathbf{T}$.

2. The following subspaces of $\mathbb{R}^3$

$$\begin{aligned} \mathbf{S}_1 &= \{\mathbf{u} = (\gamma, \gamma, \delta); \quad \gamma, \delta \in \mathbb{R}\}, \\ \mathbf{S}_2 &= \{\mathbf{v} = (\alpha, \beta, \alpha); \quad \alpha, \beta \in \mathbb{R}\} \end{aligned}$$

   are such that $\mathbf{S}_1 \cap \mathbf{S}_2 = \{\mathbf{w} = (\alpha, \alpha, \alpha); \quad \alpha \in \mathbb{R}\}$ and $\mathbf{S}_1 + \mathbf{S}_2 = \mathbb{R}^3$, since any vector (x, y, z) of $\mathbb{R}^3$ can be written as $(x, y, z) = (\frac{x}{2}, \frac{x}{2}, z - \frac{x}{2}) + (\frac{x}{2}, y - \frac{x}{2}, \frac{x}{2})$.

1.4 The linear span

We define the subspace *spanned* (i.e. *generated*) by a finite set $A = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r\}$ of vectors of $\mathbb{R}^n$, which we denote as $\mathbf{span}(A)$: $\mathbf{span}(A)$ is the set of all linear combinations of vectors of A:

$$\mathbf{span}(A) = \{\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_r \mathbf{a}_r, \quad \lambda_i \in \mathbb{R}, i = 1, \ldots, r\}.$$

This set is closed with respect to addition and scalar multiplication in $\mathbb{R}^n$, since given any two linear combinations $\mathbf{v} = \sum_{i=1}^r \lambda_i \mathbf{a}_i$ and $\mathbf{w} = \sum_{i=1}^r \mu_i \mathbf{a}_i$

$$\mathbf{v} + \mathbf{w} = \sum_{i=1}^{r} \lambda_i \mathbf{a}_i + \sum_{i=1}^{r} \mu_i \mathbf{a}_i = \sum_{i=1}^{r} (\lambda_i + \mu_i)\mathbf{a}_i$$

which is still a linear combination of $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r\}$ so it still belongs to $\mathbf{span}(A)$. Similarly, for all $\alpha \in \mathbb{R}$:

$$\alpha \sum_{i=1}^{r} \lambda_i \mathbf{a}_i = \sum_{i=1}^{r} (\alpha\lambda_i)\mathbf{a}_i$$

which is a vector of $\mathbf{span}(A)$. Thus $\mathbf{span}(A)$ is a subspace of $\mathbb{R}^n$.

Example 1.7.

1. In $\mathbb{R}^3$ the set $\{\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0)\}$ spans the subspace

$$\mathbf{span}(\mathbf{e}_1, \mathbf{e}_2) = \{\lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 = (\lambda_1, \lambda_2, 0), \quad \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

2. In $\mathbb{R}^2$, the vector $\mathbf{v} = (1, -1)$ spans the subspace:

$$\mathbf{span}(\mathbf{v}) = \{\lambda\mathbf{v} = (\lambda, -\lambda), \quad \lambda \in \mathbb{R}\}$$

   which geometrically consists of all the vectors belonging to the line $y = -x$.

3. The null subspace $\{\mathbf{0}\}$ of $\mathbb{R}^n$is spanned by the null vector $\mathbf{0}$.

4. The whole space $\mathbb{R}^n$ is spanned by the set:

$$E = \{\mathbf{e}_1 = (1, 0, \cdots, 0), \mathbf{e}_2 = (0, 1, \cdots, 0), \cdots, \mathbf{e}_n = (0, 0, \ldots, 1)\}$$

   since

$$\mathbf{v} = (x_1, x_2, \ldots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

The following is an important property of $\mathbb{R}^n$ and of all its subspaces.

Proposition 1.5. *Every subspace* $\mathbf{S} \subseteq \mathbb{R}^n$ *is* finitely generated*, i.e. all the vectors of* $\mathbf{S}$ *can be expressed as linear combinations of a* ***finite number*** *of vectors of* $\mathbf{S}$*.*

This is not true of all vector spaces: the one consisting of all the real functions in NOT finitely generated.

1.5 Bases and dimension of subspaces of $\mathbb{R}^n$

Suppose $\mathbf{S}$ is a subspace of $\mathbb{R}^n$ spanned by the set $A = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r\}$: $\mathbf{S} = \mathbf{span}(A)$. We may ask ourselves whether all the vectors of A are essential in order to span $\mathbf{S}$. This is related to whether or not they are dependent.

Theorem 1.1. *Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r\} \subset \mathbb{R}^n$, and let B be a linearly independent maximal subset of A. Then A and B span the same subspace.*

Definition 1.8. Let $\mathbf{S} \subseteq \mathbb{R}^n$ be a non-null subspace of $\mathbb{R}^n$. We say that the set of vectors $B = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r\} \subseteq \mathbf{S}$ is a *basis* of $\mathbf{S}$ if:

- B spans $\mathbf{S}$: $\mathbf{S} = \mathbf{span}(B)$;
- the vectors of B are linearly independent.

In other words, a set of vectors $B = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r\}$ forms a basis of a subspace $\mathbf{S}$ if and only if all the vectors of $\mathbf{S}$ can be written in a unique way as linear combinations of the vectors $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r$..

Example 1.8.

1. The space $\mathbb{R}^3$ is spanned by

$$E = \{\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1)\},$$

This set is not redundant;the exclusion of one generator, e.g. $\mathbf{e}_3$, gives the set $F = \{\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0)\}$ and

$$\begin{aligned} \mathbf{span}(F) &= \{\alpha\mathbf{e}_1 + \beta\mathbf{e}_2 = \alpha(0,1,0) + \beta(0,1,0); \alpha, \beta \in \mathbb{R}\} \\ &= \{(\alpha, \beta, 0); \alpha, \beta \in \mathbb{R}\} \end{aligned}$$

which is a subspace of $\mathbb{R}^3$ that does not contain all the vectors of the space $\mathbb{R}^3$; for instance the vector $\mathbf{e}_3$ and all its multiples are missing.

More in general, in $\mathbb{R}^n$ the set

$$E = \{\mathbf{e}_1 = (1,0,\cdots,0), \mathbf{e}_2 = (0,1,\cdots,0), \cdots, \mathbf{e}_n = (0,0,\cdots,1)\}$$

spans the whole space $\mathbb{R}^n$ and is linearly independent, thus it is a basis of the vector space $\mathbb{R}^n$, called the *canonical* or *standard* basis of $\mathbb{R}^n$.

2. In $\mathbb{R}^4$ the vectors

$$\mathbf{a}_1 = (1,0,1,1), \mathbf{a}_2 = (0,1,1,1), \mathbf{a}_3 = (2,1,1,-1)$$

are linearly independent, hence they form a basis of the subspace spanned by them $\mathbf{S} = \mathbf{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

3. In $\mathbb{R}^3$ the subspace $\mathbf{T}$ spanned by

$$\mathbf{a}_1 = (1,0,1), \mathbf{a}_2 = (2,0,2), \mathbf{a}_3 = (-1,0,-1), \mathbf{a}_4 = (0,0,0)$$

has a basis consisting of just one vector, for instance

$$B = \{\mathbf{a}_1 = (1,0,1)\}.$$

Another basis of $\mathbf{T}$ is $\{\mathbf{a}_2\}$, and so is $\{\mathbf{a}_3\}$, which shows that bases of subspaces are not unique.

Remark
Every non-null subspace $\mathbf{S}$ of $\mathbb{R}^n$ (including $\mathbb{R}^n$ itself), has a basis.

A fundamental property is the following.

Theorem 1.2. *Two different bases of the same subspace* $\mathbf{S}$ *always have the same number of elements.*

Definition 1.9. The number of vectors forming a basis of a subspace $\mathbf{S}$ is called the *dimension* of $\mathbf{S}$, denoted $\dim(\mathbf{S})$ or $\dim \mathbf{S}$. The dimension of the null subspace is taken to be zero.

Thus the dimension of a vector subspace $\mathbf{S} \subseteq \mathbb{R}^n$ is the **maximum** number of independent vectors in $\mathbf{S}$. The dimension of a subspace of $\mathbb{R}^n$ is also the **minimum** number of vectors in a spanning set.

Remark
If $\mathbf{S}$ is a subspace of $\mathbb{R}^n$ of dimension r, then $0 \leq r \leq n$.

Going back to the previous examples:

1. The vector space $\mathbb{R}^n$ has dimension n, since the standard basis:

   $$E = \{\mathbf{e}_1 = (1, 0, \cdots, 0), \mathbf{e}_2 = (0, 1, \cdots, 0), \cdots, \mathbf{e}_n = (0, 0, \cdots, 1)\}$$

   has n elements.

2. The subspace $\mathbf{S}$ of $\mathbb{R}^4$ spanned by the vectors:

   $$\mathbf{b}_1 = (1, 0, 1, 1), \mathbf{b}_2 = (0, 1, 1, 1), \mathbf{b}_3 = (2, 1, 1, -1)$$

   has dimension 3, since these vectors are independent and thus form a basis of $\mathbf{S}$.

3. The subspace $\mathbf{T}$ of $\mathbb{R}^3$ spanned by

   $$\mathbf{a}_1 = (1, 0, 1), \mathbf{a}_2 = (2, 0, 2), \mathbf{a}_3 = (-1, 0, -1), \mathbf{a}_4 = (0, 0, 0)$$

   has dimension 1, i.e. all its bases consist of just one vector.

Example 1.9.

4. We want to find out whether the vectors

   $$\mathbf{s}_1 = (1, 1, 0), \mathbf{s}_2 = (-1, 0, 1), \mathbf{s}_3 = (2, 3, 0)$$

   are a basis of $\mathbb{R}^3$. It suffices to check that they are linearly independent.

5. Is the set

   $$K = \{\mathbf{v}_1 = (1, 1, 1, 0), \mathbf{v}_2 = (0, 1, 0, -1), \mathbf{v}_3 = (1, 2, 0, 1), \mathbf{v}_4 = (1, 2, 1, -1)\}$$

a basis of $\mathbb{R}^4$? Since we have four vectors in a 4-dimensional space, they form a basis of $\mathbb{R}^4$ if they are linearly independent. It is easy to check that

$$\mathbf{v}_4 = \mathbf{v}_1 + \mathbf{v}_2 + 0\mathbf{v}_3$$

thus K *is not* a basis of $\mathbb{R}^4$. However, K spans a subspace, $\mathbf{W} = \mathbf{span}(K)$. What is the dimension of $\mathbf{W}$? We look for a maximal independent subset of K. For instance

$$K^* = \{\mathbf{v}_1 = (1,1,1,0), \mathbf{v}_2 = (0,1,0,-1), \mathbf{v}_3 = (1,2,0,1)\}.$$

Then $\mathbf{W} = \mathbf{span}(K) = \mathbf{span}(K^*)$, and thus $\mathbf{W}$ has dimension 3.

Remark

The vector space $\mathbb{R}^n$ contains subspaces of any dimension r, with $r = 0, 1, \ldots, n-1, n$. For $r = 0$ take the null subspace. Otherwise, if $r \geq 1$, starting from the canonical basis of $\mathbb{R}^n$,

$$E = \{\mathbf{e}_1 = (1,0,\cdots,0), \mathbf{e}_2 = (0,1,\cdots,0), \cdots, \mathbf{e}_n = (0,0,\cdots,1)\},$$

we can build subspaces of any dimension $1, 2, \ldots, n-1, n$. We take $\mathbf{span}(\{\mathbf{e}_1\})$, which has dimension 1, $\mathbf{span}(\{\mathbf{e}_1, \mathbf{e}_2\})$ with dimension 2, $\ldots$, $\mathbf{span}(\{\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}\})$ with dimension $n-1$, while $\mathbf{span}(E) = \mathbb{R}^n$ has dimension n.

One-dimensional subspaces of $\mathbb{R}^n$ are called *lines through the origin*, 2-dimensional subspaces are called *planes through the origin*, the $(n-1)$-dimensional ones are the *hyperplanes through the origin* of $\mathbb{R}^n$.

Remark

Given two subspaces $\mathbf{S}, \mathbf{T}$ of $\mathbb{R}^n$ such that:

$$\mathbf{S} \subseteq \mathbf{T}$$

and

$$\dim \mathbf{S} = \dim \mathbf{T},$$

then:

$$\mathbf{S} = \mathbf{T}.$$

This is due to the fact that any basis of $\mathbf{S}$ is also a basis of $\mathbf{T}$, hence the two subspaces are spanned by the same set.

The following remarkable result holds true:

Theorem 1.3 (Grassmann's Theorem)**.** *Let* $\mathbf{S}$ *and* $\mathbf{T}$ *be two subspaces of* $\mathbb{R}^n$ *then*

$$\dim(\mathbf{S} + \mathbf{T}) + \dim(\mathbf{S} \cap \mathbf{T}) = \dim(\mathbf{S}) + \dim(\mathbf{T}).$$

Chapter 2

2 Matrices

2.1 The space $\mathbb{R}^{m\times n}$

Given the natural numbers m and n, $(m, n \geq 1)$, consider mn real numbers, displayed in an $m \times n$ array

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

This is referred to as a *matrix with m rows and n columns*, also described as an $m \times n$ matrix, with elements (or *entries*) in $\mathbb{R}$. An alternative notation uses square brackets instead of round ones

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Each entry of the matrix is identified by two subscripts, the first one indicates the row and the second one the column: entry a_{ij} is the one at the intersection of row i with column j. For instance

$$\begin{bmatrix} 3 & 4 & 2 & 0 \\ 1 & 0 & 1 & \sqrt{2} \end{bmatrix}$$

is a 2×4 matrix where $a_{13} = 2$. A matrix A can also be denoted as

$$A = [a_{ij}], \qquad i = 1, \dots, m, \quad j = 1, \dots, n$$

for short.

For fixed m ed n, we denote with

$$\mathbb{R}^{m\times n} = \{A = [a_{ij}] \mid a_{ij} \in \mathbb{R}, i = 1, \dots, m,\ j = 1, \dots, n, \}.$$

the set of all the matrices with m rows and n columns with $\mathbb{R}$ coefficients. Two matrices $A = [a_{ij}], B = [a_{ij}] \in \mathbb{R}^{m\times n}$ are *equal* if $a_{ij} = b_{ij}$, for each $i = 1, \dots, m$ and $j = 1, \dots, n$.

When $m = n$ the matrix is called a *square matrix,* otherwise a *rectangular matrix.* The number of row (that equals the number of columns) of a square

matrix is called the *order* of the matrix. In a square matrix of order n, the $n-$tuple $(a_{11}, a_{22}, \ldots, a_{nn})$ is called the *main diagonal* of A

$$A = \begin{bmatrix} \boxed{a_{11}} & a_{12} & \ldots & a_{1n} \\ a_{21} & \boxed{a_{22}} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & \boxed{a_{nn}} \end{bmatrix}.$$

Given a square matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, we say that A is

- *upper triangular* if all the entries below the main diagonal are zero, that is $a_{ij} = 0$ if $i > j$;
- *lower triangular* if all the entries above the main diagonal are zero, that is $a_{ij} = 0$ if $i < j$
- *diagonal* if all the entries off the main diagonal are zero, that is $a_{ij} = 0$ if $i \neq j$ (notice that a diagonal matrix is both upper and lower triangular).

$$U = \begin{bmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ 0 & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u_{nn} \end{bmatrix} \quad L = \begin{bmatrix} l_{11} & 0 & \ldots & 0 \\ l_{21} & l_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \ldots & l_{nn} \end{bmatrix} \quad D = \begin{bmatrix} d_{11} & 0 & \ldots & 0 \\ 0 & d_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_{nn} \end{bmatrix}$$

Example 2.1. 1.

$$A = \begin{bmatrix} 3 & 4 & 2 & 0 \\ 0 & 0 & 1 & \sqrt{2} \\ 1 & 2 & 0 & 4 \end{bmatrix}$$

is a matrix with 3 rows and 4 columns.

2. $C = \begin{bmatrix} -3 & 4 & 0 \\ -5 & 0 & \sqrt{3} \\ -1 & 0 & \pi \end{bmatrix}$ is a 3×3 square matrix and its main diagonal is the vector $(-3, 0, \pi)$.

A $1 \times n$ matrix, i.e. a matrix with just one row, is called a *row matrix*. For instance

$$K = \begin{bmatrix} 1 & 0 & -3 & 0 & 2 & 9 \end{bmatrix},$$

which has 1 row and 6 columns, is a row matrix. Similarly, an $m \times 1$ matrix, i.e. a matrix with just one column, like for instance

$$H = \begin{bmatrix} -1 \\ -7 \\ -1 \\ 0 \\ -\sqrt{5} \end{bmatrix}$$

with 5 rows and 1 column, is called a *column matrix*. Row and column matrices can be identified with the corresponding vectors e.g. matrix K can be identified with vector $(1, 0, -3, 0, 2, 9)$ while matrix H can be identificated with vector $(-1, -7, -1, 0, -\sqrt{5})$. In other words, we have a natural bijection between the sets $\mathbb{R}^n$, $\mathbb{R}^{n\times 1}$ and $\mathbb{R}^{1\times n}$.

2.2 The rank of a matrix

Let $A \in \mathbb{R}^{m\times n}$ be a matrix with m rows and n columns. The rows $\mathbf{r}_1, \ldots, \mathbf{r}_m$ of A are m vectors of the vector space $\mathbb{R}^n$. This set spans a subspace of $\mathbb{R}^n$ called the *row space* of matrix A indicated as $\mathcal{R}(A)$. The dimension of $\mathcal{R}(A)$ is called the *row rank* of A and is, by definition, the maximum number of independent rows of A.

Clearly this dimension cannot be greater than the number of rows, i.e. m, nor can it be greater than n, since $\mathcal{R}(A) \subseteq \mathbb{R}^n$ thus

$$0 \leq \dim \mathcal{R}(A) \leq \min\,(m, n).$$

The same applies to the columns of A. They are n vectors $\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_n \in \mathbb{R}^m$. The subspace spanned by the columns of A is called the *column space* of A, indicated as $\mathcal{C}(A)$. The *column rank* of A, namely the dimension of $\mathcal{C}(A)$, is such that

$$0 \leq \dim\, \mathcal{C}(A) \leq \min(m, n).$$

Example 2.2. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 3 \\ -1 & 4 & -3 & 6 \end{bmatrix} \in \mathbb{R}^{3\times 4},$$

the rows space of A is the subspace

$$\mathcal{R}(A) = \mathbf{span}\,\{\mathbf{r}_1 = (1, 0, 1, 0), \mathbf{r}_2 = (0, 2, -1, 3), \mathbf{r}_3 = (-1, 4, -3, 6)\} \subseteq \mathbb{R}^4.$$

The rows are linearly dependent since $\mathbf{r}_3 = -\mathbf{r}_1 + 2\mathbf{r}_2$. A basis of $\mathcal{R}(A)$ is thus $\{\mathbf{r}_1, \mathbf{r}_2\}$ since $\mathbf{r}_1$ and $\mathbf{r}_2$ are not proportional. So the row rank of A, that is $\dim \mathcal{R}(A)$, is 2.

The columns of matrix A are vectors of $\mathbb{R}^3$ and

$$\mathcal{C}(A) = \mathbf{span}\left\{\mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \mathbf{c}_3 = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}, \mathbf{c}_4 = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}\right\} \subseteq \mathbb{R}^3.$$

Also the columns are linearly dependent, since $\mathbf{c}_3 = \mathbf{c}_1 - \frac{1}{2}\mathbf{c}_2$ and $\mathbf{c}_4 = \frac{3}{2}\mathbf{c}_2$. So

$$\mathcal{C}(A) = \mathbf{span}(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4) = \mathbf{span}(\mathbf{c}_1, \mathbf{c}_2).$$

So $\{\mathbf{c}_1, \mathbf{c}_2\}$ is a basis for $\mathcal{C}(A)$, since $\mathbf{c}_1, \mathbf{c}_2$ are linearly independent, and also the column rank of A is 2.

In this example the row rank and the column rank of A coincide. **This is in fact a general result**, in any matrix the maximum number of linearly independent rows equals the maximum number of linearly independent columns.

Theorem 2.1. *Let $A \in \mathbb{R}^{m\times n}$, and let $\mathcal{C}(A) \subseteq \mathbb{R}^m$ and $\mathcal{R}(A) \subseteq \mathbb{R}^n$ be the column space and the row space of A respectively, then*

$$\dim \mathcal{C}(A) = \dim \mathcal{R}(A)$$

Since the row rank of every matrix A is the same as the column rank, we will simply refer to it as the *rank* of the matrix A. That is

$$\operatorname{rank}(A) = \dim \mathcal{R}(A) = \dim \mathcal{C}(A).$$

Clearly, if $A \in \mathbb{R}^{m\times n}$ then

$$0 \leq \operatorname{rank}(A) \leq \min(m, n).$$

Finding the rank of a matrix it is not an easy problem and requires *ad hoc* methods as we will see in the next subsection.

However, for some special matrices it is easy. For instance, it is not difficult to check that for a diagonal matrix the rank is the number of non-zero elements on the main diagonal. As a consequence, for each $r = 0, 1, \ldots, n$ there exists an order n matrix of rank r.

Definition 2.1. An order n matrix $A \in \mathbb{R}^{n\times n}$ is said to be of *full rank* or *regular* or *non-singular* if it has rank n; if the rank of A is less than n we call it *singular*.

2.3 Matrix echelon form and rank computation

A row in a matrix is said to be a *zero row* if it has all zero entries (i.e. it is the zero vector).

Definition 2.2. A matrix A is said to be in *(row) echelon form* if

- all nonzero rows are above all zero ones and
- the leading entry (that is the first non zero entry from the left) of a non zero row is always strictly to the right of the leading entry of the row above it.

The leading entries of a matrix in (row) echelon form are also called *pivots*.

Example 2.3. The matrices

$$\begin{pmatrix} 0 & 0 & \boxed{2} & \pi & -1 \\ 0 & 0 & 0 & 0 & \boxed{\sqrt{7}} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \boxed{-3} & 0 & 0 & 8 & 5 & 1 \\ 0 & \boxed{\sqrt{2}} & 3 & 2 & 0 & 0 \end{pmatrix}$$

are in echelon form and the pivots are the squared entries. On the contrary, the matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & \pi & -1 \\ 0 & 0 & 0 & 0 & \sqrt{7} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \sqrt{2} & 3 & 2 & 0 & 0 \\ -3 & 0 & 0 & 8 & 5 & 1 \end{pmatrix}$$

are not in echelon form.

Clearly the number of pivots of a matrix in echelon form equals the number of nonzero rows. Moreover, for matrices in echelon form the rank is really easy to compute.

Proposition 2.1. *The rank of a matrix in echelon form equals the number of nonzero rows.*

Being in echelon form is a very special condition, but each matrix can be transformed to echelon form by operating on its rows. We call *elementary row operations* the following operations performed on the rows of a matrix

- row switching: a row is switched with another $\mathbf{r}_i \leftrightarrow \mathbf{r}_j$
- scalar multiplication: each element in a row is multiplied by a nonzero scalar $\mathbf{r}_i \leftarrow \alpha \mathbf{r}_i$
- row combination: a row is replaced by the sum of itself and the multiple of another row $\mathbf{r}_i \leftarrow \mathbf{r}_i + \alpha \mathbf{r}_j$.

We say that two matrices are *row equivalent* if it is possible to change one to another by a finite sequence of elementary row operations.

Theorem 2.2. *Each matrix can be transformed to row echelon form by means of a finite sequence of elementary row operations, that is each matrix is row equivalent to a matrix in echelon form.*

A key observation for constructing an algorithm to transform to echelon form, is that the two conditions of Definition 2.1 imply that all entries in a column below a pivot are zeros. So, starting with a fixed matrix, we can transform it to echelon form as follows

1. among the nonzero entries choose one of the leftmost and switch its row with the first one
2. combining scalar multiplication and row combination, reduce to zero all the entries below the chosen element
3. consider the matrix obtained by forgetting about the first row and repeat steps 1. and 2.

Generally there are infinite many different matrices in echelon form all equivalent to a given one. The important fact is that, since elementary operation do not change the row space, they all have the same rank.

Proposition 2.2. *If A is row equivalent to B then $\mathcal{R}(A) = \mathcal{R}(B)$ and so* $\operatorname{rank}(A) = \operatorname{rank}(B)$.

Thus, to compute the rank of a given matrix, first we use elementary operations to transform to echelon form and then we count the number of pivots of the echelon form.

In a similar fashion, it is possible to give the definition of column echelon form and elementary column operations. Moreover, all the results presented in this section still hold once we replace the word "row" with the word "column".

2.4 Vector structure of $\mathbb{R}^{m\times n}$

We define an operation of addition of two matrices, and another operation of multiplication of a matrix by a scalar.

2.4.1 Sum of two matrices

In the set $\mathbb{R}^{m\times n}$ of all $m \times n$ matrices the *sum* of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is defined *entrywise*, i.e.

$$A + B = [a_{ij} + b_{ij}],$$

namely the sum of A and B is the matrix whose entries are the sums of the corresponding entries in A and B.

Example 2.4. If $A = \begin{bmatrix} 1 & 0 \\ 7 & 3 \\ -1 & 4 \\ 0 & -2 \\ \sqrt{5} & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & -5 \\ -1 & 1 \\ -7 & -3 \\ 0 & \sqrt{3} \end{bmatrix}$, their sum is the matrix

$$C = A + B = \begin{bmatrix} 1 & 0 \\ 8 & -2 \\ -2 & 5 \\ -7 & -5 \\ \sqrt{5} & \sqrt{3} \end{bmatrix}.$$

Properties of the matrix sum in $\mathbb{R}^{m\times n}$:

Commutative property

$$A + B = B + A$$

for all $A, B \in \mathbb{R}^{m\times n}$.

Associative property

$$(A + B) + C = A + (B + C)$$

for all $A, B, C \in \mathbb{R}^{m\times n}$.

Identity element The matrix with all its entries equal to zero is called the *null* matrix and is denoted as $O_{m\times n}$; the null matrix has no effect on the sum, i.e.

$$A + O_{m\times n} = O_{m\times n} + A = A$$

for all $A \in \mathbb{R}^{m\times n}$.

Opposite elements For all $A = [a_{ij}]$ the matrix $-A = [-a_{ij}]$ is such that

$$A + (-A) = (-A) + A = O_{m\times n}.$$

2.4.2 Scalar multiplication

We can also multiply a matrix by a real number: if $A = [a_{ij}]$ and $\alpha \in \mathbb{R}$

$$\alpha A = [\alpha a_{ij}]$$

and the following properties hold true:

$$\alpha(A + B) = \alpha A + \alpha B$$

$$(\alpha + \beta)A = \alpha A + \beta A$$

$$\alpha(\beta A) = (\alpha\beta)A$$

$$1A = A$$

for all $A, B \in \mathbb{R}^{m\times n}$ and for all $\alpha, \beta \in \mathbb{R}$.

In conclusion, the set of all matrices $\mathbb{R}^{m\times n}$ is a vector space on the real numbers. The terminology of Chapter 1 can be used for matrices as well, e.g. linear combinations of matrices, linearly independent matrices, spans, subspaces of $\mathbb{R}^{m\times n}$ and so on.

Example 2.5.

1. $$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

2. In $\mathbb{R}^{2\times 2}$ the linear combination of the matrices

   $$A = \begin{bmatrix} 1 & -1 \\ -7 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$$

   with weights $\alpha = 3, \beta = -2, \gamma = 1$ is

   $$3A - 2B + C = \begin{bmatrix} 2 & -5 \\ -19 & 10 \end{bmatrix}.$$

3. In $\mathbb{R}^{n\times n}$ the matrices with only one element equal to 1 and all the others equal to 0 are linearly independent and they span the whole space $\mathbb{R}^{n\times n}$. For instance, $\mathbb{R}^{2\times 2}$ is spanned by the matrices

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

since

$$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} = a_{11}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + a_{12}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + a_{21}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} + a_{22}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

More in general $\mathbb{R}^{m\times n}$ is spanned by the set of mn matrices having just one element equal to 1 and all the others equal to 0, which are clearly linearly independent. Thus the dimension of $\mathbb{R}^{m\times n}$ is mn.

4. The set $\mathcal{D}$ of all $n\times n$ diagonal matrices is a **subspace** of the vector space $\mathbb{R}^{n\times n}$. $\mathcal{D}$ is spanned by the linearly independent matrices

$$E_1 = \begin{bmatrix}1 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & 0 & \vdots\\ 0 & 0 & \dots & 0\end{bmatrix}, E_2 = \begin{bmatrix}0 & 0 & \dots & 0\\ 0 & 1 & \dots & 0\\ \vdots & \vdots & 0 & \vdots\\ 0 & 0 & \dots & 0\end{bmatrix}, ..., E_n = \begin{bmatrix}0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & 0 & \vdots\\ 0 & 0 & \dots & 1\end{bmatrix}$$

since for all $d_{11}, d_{22}, ..., d_{nn} \in \mathbb{R}$,

$$D = \begin{bmatrix}d_{11} & 0 & \dots & 0\\ 0 & d_{22} & \dots & 0\\ \vdots & \vdots & 0 & \vdots\\ 0 & 0 & \dots & d_{nn}\end{bmatrix} = d_{11}E_1 + d_{22}E_2 ... + d_{nn}E_n$$

so the dimension of $\mathcal{D}$ is n.

2.5 Matrix product

As we will see, one of the main feature of matrix theory is the possibility of identifying an $m\times n$ matrix A it with a (linear) function from the vector space $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$. Given two matrices $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B = [b_{jk}] \in \mathbb{R}^{n\times p}$, we can think of B as a function from the vector space $\mathbb{R}^p$ to the vector space $\mathbb{R}^n$. Now think of applying both functions in succession, namely vectors go from $\mathbb{R}^p$ to $\mathbb{R}^n$ by means of B, and then from $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$ by means of A. Can we do this in one step instead of two, by means of just one matrix C? If the answer is yes, C will have to be an $m\times p$ matrix.

It is indeed possible to define such a matrix C, which is called the *row-by-column product* of A and B, and is denoted

$$C = AB.$$

The definition of C is based on the definition of *dot* or *inner product* of two $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ vectors of $\mathbb{R}^n$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^{n} x_j y_j.$$

The rows of A have n components, i.e. are vectors of $\mathbb{R}^n$, and so are the columns of B. Then we can take the dot product of each row of A by each column of B. In detail, the *dot* product of $\mathbf{a}^i$ the row i of A by $\mathbf{b}_k$, the column k of B, where $i = 1, \ldots, m$ and $k = 1, \ldots, p$, is

$$c_{ik} = \langle \mathbf{a}^i, \mathbf{b}_k \rangle = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

The matrix $C = (c_{ik}) \in \mathbb{R}^{m \times p}$ is the row-by-column product AB.

Example 2.6.

1. Given the matrices $A = \begin{bmatrix} 1 & -3 & 0 \\ 1 & -1 & 2 \end{bmatrix} \in \mathbb{R}^{2\times 3}$ and $B = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}$, the row-by-column product of A by B is $AB = \begin{bmatrix} 3 & 3 & -2 \\ 5 & 1 & 2 \end{bmatrix} \in \mathbb{R}^{2\times 3}$.

   Note that the row-by-column product BA is undefined, since the rows of B are vectors of $\mathbb{R}^3$ while the columns of A are vectors of $\mathbb{R}^2$.

2. The row-by-column product of $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ by $B = \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 0 & 4 \end{bmatrix}$ is doable. Taking the inner product of each row of A by each column of B, the result is:

$$C = AB = \begin{bmatrix} 0 & -2 \\ 1 & 15 \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

   In this case it is also possible to multiply B by A, since the rows of B and the columns of A are vectors of $\mathbb{R}^2$:

$$BA = \begin{bmatrix} 4 & -2 & 6 \\ 1 & -1 & 4 \\ 8 & -4 & 12 \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

   The two products are clearly very different!

3. Now take the matrices $H = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ and $K = \begin{bmatrix} 2 & 1 \\ -5 & 2 \end{bmatrix}$; we can compute both products HK and KH:

$$\begin{aligned} HK &= \begin{bmatrix} -5 & 2 \\ -12 & 3 \end{bmatrix} \\ KH &= \begin{bmatrix} -1 & 4 \\ -2 & -1 \end{bmatrix} \end{aligned}$$

   and $HK \neq KH$.

The three examples above show that the row-column product of two matrices A and B in general does not satisfy the commutative property. It may so happen, however, that $AB = BA$ for some particular pairs of matrices A and B; in which case we say that the matrices *commute*.

For instance, for $D_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, D_2 = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ we get

$$D_1 D_2 = D_2 D_1 = \begin{bmatrix} -2 & 0 \\ 0 & 12 \end{bmatrix}.$$

This is in fact a general result: all diagonal matrices of the same size commute, since given

$$D_1 = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}, D_2 = \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n \end{bmatrix}$$

then

$$D_1 D_2 = \begin{bmatrix} a_1 b_1 & 0 & \dots & 0 \\ 0 & a_2 b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n b_n \end{bmatrix} = \begin{bmatrix} b_1 a_1 & 0 & \dots & 0 \\ 0 & b_2 a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n a_n \end{bmatrix} = D_2 D_1.$$

2.6 Properties of the matrix product

Associative property

Suppose A is $m \times n$, B is $n \times s$ and C is $s \times t$. Then AB is $m \times s$ and BC is $n \times t$, so both products (AB) by C and A by (BC) are possible, both giving rise to an $m \times t$ matrix. The following property holds

$$(AB)C = A(BC)$$

Left distributive property

Suppose A is $m \times n$, B and C are both $n \times s$. Then $B + C$ is $n \times t$, so the products A by B, A by C, and A by $(B + C)$ are all possible, all giving rise to $m \times s$ matrices. The following property holds

$$A(B + C) = AB + AC,$$

Right distributive property

Suppose A and B are both $m \times n$, and C is $n \times s$. Then $A + B$ is $m \times n$, so the products A by C, B by C, and $(A + B)$ by C are all possible, all giving rise to $m \times s$ matrices. The following property holds

$$(A + B)C = AC + BC$$

2.7 The transpose of a matrix

Definition 2.3. For every matrix $A \in \mathbb{R}^{m\times n}$ we can define another matrix whose rows are equal to the columns of A and whose columns are equal to the rows of A. It is called the *transpose* of matrix A and is denoted as A^T. Note that $A^T \in \mathbb{R}^{n\times m}$.

Example 2.7. The transpose of $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 8 & 4 \\ 1 & 1 & 0 \\ -2 & 3 & -4 \end{bmatrix}$ is $A^T = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 2 & 8 & 1 & 3 \\ -3 & 4 & 0 & -4 \end{bmatrix}$.

Properties of the transpose:

For all $A, B \in \mathbb{R}^{m\times n}$ and all $\alpha \in \mathbb{R}$

1. $(A+B)^T = A^T + B^T$ (additivity)
2. $(\alpha A)^T = \alpha A^T$ (homogeneity)
3. $(A^T)^T = A$
4. $(AB)^T = B^T A^T$, for all $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times t}$, that is the transpose of the product of two matrices is the product of the transposes **in reverse order.**
5. $\text{rank}(A) = \text{rank}(A^T)$.

Example 2.8. 1. We want to find the transpose of the matrix H which is a linear combination of the matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -3 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} -2 & 1 & 1 \\ 4 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

with coefficients $\alpha = -1, \beta = 3, \gamma = 2$.

Then

$$\begin{aligned} H &= -A + 3B + 2C \\ &= \begin{bmatrix} -1 & 0 & -1 \\ 3 & -4 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 3 & 3 \\ 12 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 & 4 \\ 15 & -6 & 0 \end{bmatrix} \end{aligned}$$

and $H^T = \begin{bmatrix} -1 & 15 \\ 3 & -6 \\ 4 & 0 \end{bmatrix}$. But also

$$\begin{aligned} -A^T + 3B^T + 2C^T &= -\begin{bmatrix} 1 & -3 \\ 0 & 4 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} -2 & 4 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 15 \\ 3 & -6 \\ 4 & 0 \end{bmatrix} \end{aligned}$$

2. $A = \begin{bmatrix} 1 & 0 \\ -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix} \implies AB = \begin{bmatrix} 2 & 1 \\ 10 & -3 \end{bmatrix}$

$$\implies (AB)^T = \begin{bmatrix} 2 & 10 \\ 1 & -3 \end{bmatrix};$$

$$B^T = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}, A^T = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix} \implies B^T A^T = \begin{bmatrix} 2 & 10 \\ 1 & -3 \end{bmatrix}.$$

3. The transpose of an upper triangular matrix is a lower triangular one.

Sometimes the transpose of a square matrix is the matrix itself, i.e.

$$A = A^T.$$

In this case the elements of A which are in a symmetric position with respect to the main diagonal are all equal, i.e. $a_{12} = a_{21}$, $a_{13} = a_{31}$, $a_{23} = a_{32}$, etc.

Definition 2.4. A square matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, is said to be *symmetric* if

$$a_{ij} = a_{ji}, \quad \forall i, j = 1, \ldots, n.$$

or, equivalently, if $A = A^T$. The set of all real symmetric matrices of order n is denoted by $\mathcal{S}_n(\mathbb{R})$.

Example 2.9.

$$S = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 5 & 2 \\ -3 & 2 & 9 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 2 \\ -3 & 2 & 9 \end{bmatrix}$$

S is a symmetric matrix and T is not.

All diagonal matrices are symmetric.

2.8 The rank of the product of two matrices

Another way to look at the product of two matrices is the following. Let's consider the case in which the second factor is a column matrix, that is $A\mathbf{b}$ with $A \in \mathbb{R}^{m \times n}$ and $b = (b_1, \ldots, b_n) \in \mathbb{R}^{n \times 1}$. We have

$$A\mathbf{b} = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \cdots + a_{2n}b_n \\ \vdots \\ a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mn}b_n \end{pmatrix} = b_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + b_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + b_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix},$$

so $A\mathbf{b}$ is a linear combination of the columns of A with coefficients given by the components of $\mathbf{b}$.

Given two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, we have

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_p \end{bmatrix},$$

where $\mathbf{b}_i$ denotes the column i of B. This means that the columns of AB are linear combinations of the columns of A.

Similarly if $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^{1\times n}$ and $B \in \mathbb{R}^{n\times p}$ we have that the product $\mathbf{a}B$ is a linear combination of the rows of B with coefficients given by the components of $\mathbf{a}$. In the general case, if $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ then

$$AB = \begin{bmatrix} \mathbf{a}^1 B \\ \mathbf{a}^2 B \\ \vdots \\ \mathbf{a}^m B \end{bmatrix}$$

where $\mathbf{a}^i$ denotes the row i of A. So, the rows of the product are linear combinations of the rows of B.

From these observations on the row-by-column product we get the following result.

Theorem 2.3. *Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ then*

$$\text{rank}(AB) \leq \min(\text{rank}A, \text{rank}B).$$

Proof Since the columns of AB are linear combinations of the columns of A we have

$$\mathcal{C}(AB) \subseteq \mathcal{C}(A) \subseteq \mathbb{R}^m.$$

Similarly, by observing that the rows of the product are linear combinations of the rows of B we get

$$\mathcal{R}(AB) \subseteq \mathcal{R}(B) \subseteq \mathbb{R}^p.$$

The consequences in terms of dimensions are

$$\dim \mathcal{C}(AB) \leq \dim \mathcal{C}(A) \quad \Rightarrow \quad \text{rank}(AB) \leq \text{rank}A$$

and

$$\dim \mathcal{R}(AB) \leq \dim \mathcal{R}(B) \Rightarrow \quad \text{rank}(AB) \leq \text{rank}B$$

In conclusion $\text{rank}(AB) \leq \min(\text{rank}A, \text{rank}B)$. ■

Chapter 3

3 Square matrices and determinant

3.1 The ring structure of $\mathbb{R}^{n\times n}$

The comparison between properties of the algebraic operations defined on real numbers and the ones defined on matrices becomes more easy and fruitful if we set in the space of square matrices. This is due to the fact that $\mathbb{R}^{n\times n}$ is *closed* both under sum and product: the product (as well as the sum) of two square matrices of the same order is always doable and the result is still a square matrix of the same order.

In Chapter 2 we saw that the addition between matrices has the same properties owned by addition of real numbers. What about multiplication? We already saw that, in row-by-column product, the associative property holds while the commutative one does not. It is natural to ask whether the identity element and inverse elements exist or not. The first answer is easy to give, while the second will require some work.

A special (diagonal) matrix is the $n \times n$ matrix with all ones on the main diagonal and all zeroes elsewhere

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

It is called the *identity matrix* [2] of order n, since

$$AI_n = A = I_n A$$

for all $A \in \mathbb{R}^{n\times n}$.

Regarding inverse, the answer is more complicated. In $\mathbb{R}$ each non zero element is *invertible*, i.e. it has inverse with respect to multiplication. Similarly we will say that a matrix $A \in \mathbb{R}^{n\times n}$ is *invertible* if there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that

$$AB = BA = I_n.$$

It is not difficult to prove that if such a matrix exists then it is unique. So we will say that B is *the inverse* of A and denote it with A^{-1}. For instance

[2] For a rectangular matrix $A \in \mathbb{R}^{m\times n}$ we have $AI_n = A$ and $I_m A = A$. So we have a *left inverse* and a *right inverse*, but they are different.

$A^{-1} = \begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$ is the inverse of matrix $A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$ since

$$\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$
$$= \frac{1}{7}\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$
$$= \frac{1}{7}\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Clearly the null matrix O_n is not invertible since it is easy to prove that

$$AO_n = O_n A = O_n$$

for all $A \in \mathbb{R}^{n\times n}$. However, the zero matrix is not the only one. For example the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ has no inverse. To see this, let $B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ be a possible inverse. Then

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x+z & y+t \\ 2x+2z & 2y+2t \end{bmatrix} \Rightarrow \begin{cases} x+z=1 \\ y+t=0 \\ 2x+2z=0 \\ 2y+2t=1 \end{cases},$$

which is impossible.

A question arises naturally: how can we tell if a square matrix is invertible? And if it is, how do we find its inverse? The following theorem answers the first question. We postpone the second problem till after introducing determinants.

Theorem 3.1. *Let $A \in \mathbb{R}^{n\times n}$, then A is invertible if and only if A is non-singular, i.e.* $\text{rank}A = n$.

Proof If A is invertible, i.e. A^{-1} exists, then from $AA^{-1} = I_n$ and by Theorem 2.3 there follows

$$n = \text{rank}I_n = \text{rank}(AA^{-1}) \leq \min(\text{rank}A, \text{rank}A^{-1}) \leq \text{rank}(A)$$

hence $\text{rank}(A)$ (and also $\text{rank}(A^{-1})$) is at least n. But since $A \in \mathbb{R}^{n\times n}$, the conclusion is that $\text{rank}(A) = n$.
Conversely, if $\text{rank}A = n$, the columns $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ of A are a basis of $\mathbb{R}^n$. Thus every vector of $\mathbb{R}^n$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$; in particular so are the vectors of the standard basis. We write:

$$\begin{aligned} \mathbf{e}_j &= b_{1j}\mathbf{a}_1 + b_{2j}\mathbf{a}_2 + \ldots + b_{nj}\mathbf{a}_n \\ &= \sum_{k=1}^{n} b_{kj}\mathbf{a}_k \qquad \forall j = 1, 2, ..., n \end{aligned}$$

namely

$$\delta_{ij} = \sum_{k=1}^{n} b_{kj}a_{ik} = \sum_{k=1}^{n} a_{ik}b_{kj} \qquad \forall i, j = 1, 2, ..., n$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Define the matrix $B = (b_{ij})$. It is easy to check that the above set of equalities can also be written in matrix form as

$$I_n = AB.$$

By the same argument applied to the rows instead of the columns of A, we can show the existence of matrix C such that

$$I_n = CA.$$

Then $C = CI_n = CAB = I_nB = B$ hence $C = B = A^{-1}$. ■

Example 3.1. 1. In $\mathbb{R}^{2\times 2}$ consider the matrices

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The matrix A is of full rank, hence it is invertible, while B is not invertible since it has rank 1.

2. A diagonal matrix is non-singular and so invertible if and only if it has no zero elements on the main diagonal. Moreover, since in the case of diagonal matrices the product becomes entrywise, it is easy to see that if $(a_1, \ldots, a_n)$ is the main diagonal of A then also A^{-1} is diagonal and its main diagonal is $(a_1^{-1}, \ldots, a_n^{-1})$.

The set of all invertible matrices of order n is denoted with $GL_n(\mathbb{R})$ and is called *linear group of order n*.

Properties of invertible matrices

We end this section by stating some properties of invertible matrices. Let $A, B \in GL_n(\mathbb{R})$ then

- $(A^{-1})^{-1} = A$ for all invertible A.
  This means that if A^{-1} is the inverse of A, then A is the inverse of A^{-1}.
- $(AB)^{-1} = B^{-1}A^{-1}$ for all invertible A and B.
  The inverse of a product is the product of the inverses **in reverse order**, since:
  $$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n.$$
- $(A^T)^{-1} = \left(A^{-1}\right)^T$. The transpose of the inverse is the inverse of the transpose. To show this we need to check that $A^T(A^{-1})^T = I_n$:
  $$A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n.$$

3.2 Definition of determinant

Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. The *determinant* of a square matrix A, denoted as $\det A$ or $|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$, is a real number associated with A that is useful in finding the inverse of an invertible matrix and in solving linear systems. The integer n is called the *order* of the determinant.

The determinant can be defined in different ways. Before we give a proper definition, we indicate how to calculate the determinant in the cases $n = 1, 2, 3$.

- If A is 1×1, namely $A = [a_{11}]$, the determinant of A is the scalar a_{11} itself:
  $$\det A = \det[a_{11}] = a_{11}.$$
- If A is 2×2, namely $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then
  $$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$
  For instance, if $A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$, then $\det A = 2 - 15 = -13$.

- If A is a 3×3 matrix, namely $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

This rule to compute determinant is called Sarrus' Rule.

For instance

$$\det \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} = 1{\cdot}1{\cdot}0 + 0{\cdot}1{\cdot}(-1) + (-2){\cdot}1{\cdot}1 - (-2){\cdot}1{\cdot}(-1) - 1{\cdot}1{\cdot}1 - 1{\cdot}0{\cdot}0 = -5.$$

There exist different ways to define the determinant of a square matrix (note that it does not exists a similar definition for rectangular ones). In this notes we define the determinant by *induction* on the order n. We start by defining the determinant of a 1×1 matrix (that is a scalar). As we saw at the beginning of the section

$$\det[a_{11}] = a_{11}.$$

When $n > 1$, we introduce the definition of the *cofactor* of the element a_{ij} of a matrix A: it is denoted by A_{ij} and it is $(-1)^{i+j}$ times the determinant of the submatrix M_{ij} of order $n-1$ (called a *minor*) obtained deleting the i-th row and the j-th column of A:

$$A_{ij} = (-1)^{i+j} \det M_{ij}.$$

For instance, if $A = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$, the cofactor of a_{21} is $A_{21} = (-1)^{2+1} \det[4] = -4$; the cofactor of a_{11} is $A_{11} = (-1)^{1+1} \det[2] = 2$. Clearly $(-1)^{i+j}$ is always either $+1$ (if $i+j$ is even) or -1 (if $i+j$ is odd); the signs alternate according to the following scheme.

$$\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \dots & \vdots \end{bmatrix}.$$

Given $A \in \mathbb{R}^{n \times n}$, the determinant of A is defined as the sum of the elements of the first row each of them multiplied by its cofactor

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} = \sum_{j=1}^{n} a_{1j}A_{1j}.$$

In this way computing the determinant of a matrix of order n reduces to calculating n determinants of order $n-1$, and so on. The formula $\det(A) = \sum_{j=1}^{n} a_{1j}A_{1j}$ is called the *Laplace expansion* of the determinant by the first row. It is computationally very demanding as n increases!

The previous rules for the case $n = 2, 3$ are obtained directly from the general rule as shown in the following examples.

Example 3.2. 1. If $A = \begin{bmatrix} 5 & -8 \\ 2 & 3 \end{bmatrix}$, then $\det A = 5\cdot 3 - (-8)\cdot[2] = 15 - (-16) = 31$.

More in general, if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $\det A = a_{11}A_{11} + a_{12}A_{12}$ where $A_{11} = (-1)^2 \det[a_{22}] = a_{22}, \quad A_{12} = (-1)^3 \det[a_{21}] = -a_{21}$. Thus

$$\det A = a_{11}a_{22} + a_{12}(-a_{21}) = a_{11}a_{22} - a_{12}a_{21}.$$

2. If

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix},$$

then

$$\begin{aligned} \begin{vmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} &= 1A_{11} + 0A_{12} + (-2)A_{13} \\ &= 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -1 - 2\cdot 2 = -5. \end{aligned}$$

More in general, the Laplace expansion of the determinant of a 3×3 matrix by the first row is

$$\det A = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

Since $A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, A_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, A_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$,

$$\begin{aligned} \det A &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \end{aligned}$$

3.

$$\begin{vmatrix} 1 & -1 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 0 & 2 & 3 & -1 \\ 1 & 4 & 0 & -3 \end{vmatrix} = A_{11} + (-1)^2 A_{12} + 0A_{13} - 1A_{14}$$

$$= \begin{vmatrix} 3 & 1 & 0 \\ 2 & 3 & -1 \\ 4 & 0 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & -3 \end{vmatrix} - \begin{vmatrix} 2 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 4 & 0 \end{vmatrix}$$

$$= 29 - 19 + 17 = 27.$$

According to the above definition, the determinant of a square matrix is the sum of the elements of the first row each of them multiplied by its cofactor. Nevertheless, the computation of the determinant can be done using the entries of any row or any column as stated in the following result.

Theorem 3.2 (Laplace Expansion Theorem)**.** *The expansion of the determinant of a matrix A may take place by any row i or column k, (not just by the first row) and the result is the same:*

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij}.$$

$$\det(A) = a_{1k}A_{1k} + a_{2k}A_{2k} + \ldots + a_{nk}A_{nk} = \sum_{h=1}^{n} a_{hk}A_{hk}.$$

In the following example the expansion of the determinant takes place along the middle column

$$\begin{vmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$

$$= 0A_{12} + 1A_{22} + 1A_{32} = A_{22} + A_{32}$$

$$= \begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = -2 - 3 = -5.$$

The usefulness of this result is evident from the following example. The expansion of the determinant of $A = \begin{bmatrix} 2 & 1 & -2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 3 & 1 & 4 \end{bmatrix}$ by the elements of the first row requires calculating four different cofactors, i.e. four determinants of order 3; whereas the expansion of determinant of A by the first column requires just one:

$$\det A = 2A_{11} + 0A_{21} + 0A_{31} + 0A_{41} = 2A_{11}$$

with

$$A_{11} = \begin{vmatrix} 1 & -1 & 1 \\ 5 & 0 & 0 \\ 3 & 1 & 4 \end{vmatrix}.$$

The expansion of A_{11} along the second row is immediate:

$$A_{11} = \begin{vmatrix} 1 & -1 & 1 \\ 5 & 0 & 0 \\ 3 & 1 & 4 \end{vmatrix} = 5(-1)^3 \begin{vmatrix} -1 & 1 \\ 1 & 4 \end{vmatrix} = 5(-1)(-4-1) = 25.$$

Hence $\det A = 2A_{11} = 2(25) = 50$.

Consequences of the Laplace Expansion Theorem:

- The determinant of a diagonal matrix of order n

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix},$$

  is the product of the elements along the main diagonal:

$$\det D = \begin{vmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{vmatrix} = d_1 d_2 \dots d_n.$$

  To show this, it is sufficient to expand the determinant along the first row (or column), and apply induction on n.

  Clearly $\det O_n = 0$ and $\det I_n = 1$.

- Computing the determinant of an upper triangular or lower triangular matrix is just as simple. If

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

  we expand the determinant of U along the first column obtaining

$$\det U = u_{11} \begin{vmatrix} u_{22} & \dots & u_{2n} \\ \vdots & \ddots & \vdots \\ 0 & \dots & u_{nn} \end{vmatrix},$$

namely u_{11} times the determinant of a submatrix of order $n-1$ that is also upper triangular, so we repeat the procedure, until we find

$$\det U = u_{11}u_{22}\dots u_{nn},$$

again the product of the elements on the main diagonal of U. A similar result holds true for lower triangular matrices.

3.3 Properties of determinant

Here are several properties of determinants, that are often useful rules for simplifying the calculation of the determinant of a matrix.

Proposition 3.1. *For all $A \in \mathbb{R}^{n\times n}$ we have*

1. $\det A = \det A^T$.
2. *if A has one row (or one column) made of all zeroes, then* $\det A = 0$.
3. *if A contains two rows (or two columns) that are equal, then* $\det A = 0$.
4. *if A has a row (resp. column) that is the sum of two n-tuples $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and B, C denote the matrices obtained from A by replacing that row with $\mathbf{b}$, $\mathbf{c}$, we have* $\det A = \det B + \det C$.
5. *if B is a matrix obtained from A by a row (or column) switching then* $\det B = -\det A$.
6. *if B is a matrix obtained from A multiplying one row (or one column) by a scalar $\alpha \in \mathbb{R}$, then* $\det B = \alpha \det A$. *As a consequence* $\det(\alpha A) = \alpha^n \det A$.
7. *if B is a matrix obtained from A by row (or column) combination, that is adding to a row (resp. column) of A another row (resp. column) of A multiplied by a scalar, then* $\det B = \det A$.

Example 3.3.

1. If $A = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}$ and $A^T = \begin{bmatrix} 1 & 1 \\ -2 & 5 \end{bmatrix}$ then $\det A = \det A^T = 7$.
2. By property *3*, we have $\det \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} = 0$.
3. By property *4* we have $\det \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & -2 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} + \det \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.
4. Let $A = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix}$. Then

$$\det A = \begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix} = 5 \text{ and by property } 5 \ \det B = \begin{vmatrix} 0 & 5 \\ 1 & -2 \end{vmatrix} = -5.$$

5. Using property *6*

$$\det \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{5}{2} & \frac{7}{2} \end{bmatrix} = \left(\frac{1}{2}\right)^2 \det \begin{bmatrix} 1 & 3 \\ -5 & 7 \end{bmatrix} = \frac{7+15}{4} = \frac{11}{2}$$

6. If $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ multiplying the bottom row by -3 we get $B = \begin{bmatrix} 1 & -2 \\ -9 & -12 \end{bmatrix}$, then $\det A = 10$, and $\det B = -30$.

By Theorem 2.2, each matrix can be reduced by row elementary operation into a echelon form. Moreover it is easy to see that a square matrix in echelon form is upper triangular, so its determinant is the product of the elements on the main diagonal. Elementary row operations can change the determinant, but properties *5, 6, 7* explain how. Summing up, another way to compute the determinant of a matrix is the following

- use elementary row operations to transform the matrix to row echelon form
- compute the product of the elements of the main diagonal of the echelon form
- rescale the product according to the elementary operation done during the process: by an α^{-1} factor for each α scalar multiplication, by a -1 factor for each row switching.

The following theorem tells us the behavior of the determinant with respect to matrix product.

Theorem 3.3 (Binet's Theorem)**.** *If $A, B \in \mathbb{R}^{n\times n}$ then*

$$\det(AB) = \det A \det B.$$

Example 3.4*.*

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -4 \end{bmatrix}, B = \begin{bmatrix} 0 & 3 \\ 2 & -5 \end{bmatrix} \implies AB = \begin{bmatrix} 6 & -12 \\ -8 & 20 \end{bmatrix}$$

and

$$\det(AB) = 120 - 96 = 24 = (-6)(-4) = \det A \det B.$$

IMPORTANT Note however that $\det(A + B) = \det A + \det B$ **is not true**, as the following simple counterexample shows:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, A + B = \begin{bmatrix} 4 & 6 \\ 5 & 5 \end{bmatrix}$$

$\det(A + B) = -10$, while $\det A + \det B = (-1) + (-2) = -3$.

Some consequences of the Binet theorem:

1. If we denote by A^k the product of A with itself k-times we have

$$\det(A^k) = (\det A)^k.$$

2. If A is non-singular, then

$$\det(A^{-1}) = (\det A)^{-1}$$

since from $AA^{-1} = I_n$ there follows $\det(AA^{-1}) = \det A \det A^{-1} = \det I_n = 1$, and thus $\det\left(A^{-1}\right) = \dfrac{1}{\det A}$.

The determinant is also a good tool to detect whether a matrix is singular or not.

Theorem 3.4. *A square matrix A is singular if and only if* $\det A = 0$.

Proof If the $n \times n$ matrix A is singular, it means that its rank is less than n. Thus at least one of the rows of A is a linear combination of the other rows. Without loss of generality, we assume it is the bottom one, and write:

$$A = \begin{bmatrix} & & \\ & \mathbf{r}_1 & \\ & \cdots & \\ & \mathbf{r}_i & \\ & \cdots & \\ \lambda_1\mathbf{r}_1 + \cdots + \lambda_i\mathbf{r}_i + \cdots + \lambda_{n-1}\mathbf{r}_{n-1} & & \end{bmatrix}.$$

By properties *4* and *6*, we decompose the determinant of A as follows

$$\begin{aligned} \det A &= \det \begin{bmatrix} \mathbf{r}_1 \\ \cdots \\ \mathbf{r}_i \\ \cdots \\ \lambda_1\mathbf{r}_1 \end{bmatrix} + \cdots + \det \begin{bmatrix} \mathbf{r}_1 \\ \cdots \\ \mathbf{r}_i \\ \cdots \\ \lambda_i\mathbf{r}_i \end{bmatrix} + \cdots + \det \begin{bmatrix} \mathbf{r}_1 \\ \cdots \\ \mathbf{r}_i \\ \cdots \\ \lambda_{n-1}\mathbf{r}_{n-1} \end{bmatrix} \\ &= \lambda_1 0 + \cdots + \lambda_i 0 + \cdots + \lambda_{n-1} 0 = 0 \end{aligned}$$

because property *3* states that the determinant of a matrix with two equal rows is zero.

Instead of proving that $\det A = 0$ implies that A is singular we can prove the equivalent statement

$$A \text{ non singular } \Rightarrow \det A \neq 0.$$

Suppose that A is non singular then by Theorem 3.1 the matrix A is invertible that is there exists A^{-1} such that $AA^{-1} = I_n$. Applying Binet Theorem there follows $\det(AA^{-1}) = \det A \det A^{-1} = \det I_n = 1$, and thus $\det A$ should be different from zero. ■.

We end this section with a result on the rank of a matrix. Indeed, the rank of an $m \times n$ matrix can be defined in terms of determinants.

Given a matrix $A \in \mathbb{R}^{m\times n}$ a *minor* of order p of A, where $p \leq \min(m, n)$, is a $p \times p$ square submatrix obtained deleting $n - p$ rows and $n - p$ columns of A.

Theorem 3.5 (Kronecker's Theorem)**.** *Let $A \in \mathbb{R}^{m\times n}$ the following statements are equivalent*

1. $\text{rank}A = r$
2. *there exists a non singular minor of order r of A and all the minors of order greater than r are singular*
3. *there exists a non singular minor M of order r of A and all the minors of order $(r + 1)$ obtained adding one row and one column of A to M are singular.*

If M is a non singular minor of A whose order equals the rank of A we will say that M *realizes the rank* of A.

By the prevoius theorem, the rank of A, i.e. the maximum number of linearly independent rows and columns, coincides with the maximum order of minors of A that have non-zero determinant. For instance if a matrix A has 10 rows and 6 columns and its rank is 4, this means that there is at least one square submatrix of A of order 4 which is non-singular, whereas all submatrices of order 5 or higher are singular.

In conclusion, an alternative definition of the rank of a matrix is as follows: the rank of $A \in \mathbb{R}^{m\times n}$ is the maximum order of its non-singular minors. This alternative definition can be used also to compute the rank of a matrix.

3.4 Finding the inverse of a non-singular matrix

The inverse of a non-singular matrix $A \in \mathbb{R}^{n\times n}$ can be found using determinants.

We define the *adjoint* of a square matrix A as the transpose of the matrix whose elements are the cofactor of the elements of A:

$$adj(A) = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}.$$

The row-by-column product of A by its adjoint requires multiplication of each row of A by each column of $adj(A)$, namely the cofactors of the rows of A; it is possible to prove that

$$A(adjA) = \begin{bmatrix} \det A & 0 & \dots & 0 \\ 0 & \det A & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \det A \end{bmatrix}.$$

If A is non-singular, $\det A \neq 0$, so

$$A(adjA) = \det A \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = (\det A) I_n;$$

dividing both sides of the equality by $\det A \neq 0$ we get:

$$\frac{1}{\det A} A(adjA) = A\left(\frac{1}{\det A} adjA\right) = I_n$$

and $\dfrac{1}{\det A}(adjA)A = I_n$ is also true. Thus $\left(\dfrac{1}{\det A} adjA\right)$ is the inverse of A

$$A^{-1} = \frac{1}{\det A}(adjA) = \begin{bmatrix} \frac{A_{11}}{\det A} & \frac{A_{21}}{\det A} & \cdots & \frac{A_{n1}}{\det A} \\ \frac{A_{12}}{\det A} & \frac{A_{22}}{\det A} & \cdots & \frac{A_{n2}}{\det A} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{A_{1n}}{\det A} & \frac{A_{2n}}{\det A} & \cdots & \frac{A_{nn}}{\det A} \end{bmatrix}.$$

This expression gives a way of calculating the elements of A^{-1} starting from the elements of A. It may not be very easy to apply when the order n is large.

Example

let $A = \begin{bmatrix} 2 & 3 & -2 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. The cofactors of the nine elements of A are

$$\begin{aligned} A_{11} &= \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1, A_{12} = -\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = -1, A_{13} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \\ A_{21} &= -\begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} = -2, A_{22} = \begin{vmatrix} 2 & -2 \\ -1 & 0 \end{vmatrix} = -2, A_{23} = -\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = -5 \\ A_{31} &= \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} = 5, A_{32} = -\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} = -4, A_{33} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1 \end{aligned}$$

and $\det A = -3 - 2 - 2 - 2 = -9$

Thus the inverse of A is

$$A^{-1} = -\frac{1}{9}\begin{bmatrix} -1 & -2 & 5 \\ -1 & -2 & -4 \\ 2 & -5 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & \frac{2}{9} & -\frac{5}{9} \\ \frac{1}{9} & \frac{2}{9} & \frac{4}{9} \\ -\frac{2}{9} & \frac{5}{9} & \frac{1}{9} \end{bmatrix}.$$

Chapter 4

4 Linear Systems

4.1 Linear equations

A linear equation in the unknown x is for instance $3x + 2 = 0$, i.e.

$$3x = -2$$

and the solution is $x = -\frac{2}{3}$. More in general, a *linear equation* in the unknowns $x_1, x_2, \ldots, x_n$ on the real numbers is an expression of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots a_n, b \in \mathbb{R}$ are the *coefficients* of the equation and b is the *constant term.*

A *solution* of the equation is an ordered n-tuple of real numbers (i.e. a vector in $\mathbb{R}^n$) which satisfies the equality. For instance the equation

$$x_1 + 4x_2 - x_3 = 1$$

is a linear equation in the unknowns x_1, x_2, x_3; a solution is $(-2, 1, 1) \in \mathbb{R}^3$, another solution is $(1, 0, 0)$, and so on. This example is meant to stress that a solution of a linear equation in several unknowns is not just one number, but n numbers.

Clearly if $a_1 = a_2 = \cdots = a_n = b = 0$ the equation is the identity:

$$0x_1 + 0x_2 + \cdots + 0x_n = 0.$$

and every ordered n-tuple of real numbers is a solution; in other words the set of solutions is the whole space $\mathbb{R}^n$. On the other hand, if $a_1 = a_2 = \cdots = a_n = 0$ and $b \neq 0$;

$$0x_1 + 0x_2 + \cdots + 0x_n = b$$

this equation is impossible, the set of solutions is empty.

So assume at least one of the coefficients is non-zero, say $a_n \neq 0$. We rewrite the equation as

$$a_nx_n = b - a_1x_1 - a_2x_2 + \cdots - a_{n-1}x_{n-1},$$

hence dividing both sides of this equality by a_n we get:

$$x_n = a_n^{-1}b - (a_n^{-1}a_1)x_1 - (a_n^{-1}a_2)x_2 + \cdots - (a_n^{-1}a_{n-1})x_{n-1}$$

and a solution is obtained giving arbitrary values to $x_1, x_2, \ldots, x_{n-1}$ and calculating the ensuing value of x_n. For instance, a solution of the equation $x_1 + 4x_2 - x_3 = 1$ is $(4, -1, -1)$, obtained letting $x_3 = x_1 + 4x_2 - 1$, and

choosing $x_1 = 4, x_2 = -1$. But we could choose $x_1 = 0, x_2 = 1$, obtaining $x_3 = 3$.
Besides, in general there is more than one solution. To be precise, if $n > 1$ there is an infinite number of solutions. We say that the equation has ∞^{n-1} solutions. On the other hand, it is clear that not all the vectors of $\mathbb{R}^n$ are solutions.

Example 4.1. 1. The set S of the solutions of the equation $x_1 + 4x_2 - x_3 = 1$ is

$$S = \{(1 - 4x_2 + x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\} \subset \mathbb{R}^3.$$

For $x_2 = 1$ and $x_3 = 2$ we get a particular solution $(-1, 1, 2)$; the *general* solution is $(1-4x_2+x_3, x_2, x_3)$. The set S of the solutions of the equation can also be described as

$$S = \{(x_1, \frac{1}{4} - \frac{1}{4}x_1 + \frac{1}{4}x_3, x_3) \mid x_1, x_3 \in \mathbb{R}\},$$

but also as

$$S = \{(x_1, x_2, -1 + x_1 + 4x_2) \mid x_1, x_2 \in \mathbb{R}\}.$$

2. The equation $x_1 + x_2 = 0$ in $\mathbb{R}^2$ gives

$$S = \{(x_1, -x_1); x_1 \in \mathbb{R}\}$$

as the set of all possible solutions. Unlike the previous example, this is a subspace of $\mathbb{R}^2$

$$S = \{x_1(1, -1) \mid x_1 \in \mathbb{R}\} = \mathbf{span}((1, -1)).$$

3. In $\mathbb{R}^3$ we consider the same equation $x_1 + x_2 = 0$. The unknown x_3 does not appear, the coefficient of x_3 is 0; the equation has solutions

$$S = \{(x_1, -x_1, x_3); x_1, x_3 \in \mathbb{R}\},$$

so x_3 is *free* to take on all real values. Notice that also in this case S is a subspace of $\mathbb{R}^3$ since

$$S = \mathbf{span}((1, -1, 0), (0, 0, 1)).$$

Two equations in the same unknowns $x_1, x_2, \ldots, x_n$ are *equivalent* if the set of solution of the former coincides with the set of solutions of the latter. For instance, the equation $x_1 - 7x_2 + x_3 - x_4 = 1$ is equivalent to $2x_1 - 14x_2 + 2x_3 - 2x_4 = 2$. To all effects and purposes, equivalent equations are really the same equation.

4.2 Systems of linear equations

A *linear system* in the unknowns $x_1, \ldots, x_n$ on the real numbers is a set of m linear equations in the same unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}.$$

A *solution* of a linear system is an ordered real $n-$tuple $(x_1, x_2, \ldots, x_n)$, namely a real n-dimensional vector, which is a **common** solution of all the individual equations of the system. In other words, if S_i denotes the set of solutions of the i-th equation, for $i = 1, \ldots, m$, then the set of solutions S of the linear system is

$$S = S_1 \cap S_2 \cap \cdots \cap S_m.$$

We say that two linear systems in the same unknowns are *equivalent* if they have the same set of solutions.

For example, the following is a linear system

$$\begin{cases} 2x_1 - x_2 + 5x_3 + x_4 = 1 \\ -x_1 + x_2 - 3x_3 + 6x_4 = 0 \\ x_2 - x_3 + x_4 = -8 \end{cases}$$

and a possible solution is $x_1 = -\frac{17}{4}, x_2 = -\frac{35}{4}, x_3 = 0, x_4 = \frac{3}{4}$. But of course a system may also be without solutions, in other words it may be impossible to solve. For example, attempts to solve the following

$$\begin{cases} 2x_1 - x_2 = 1 \\ -x_1 + x_2 = 0 \\ x_1 - 3x_2 = -8 \end{cases}$$

lead nowhere.

A system of m linear equations in n unknowns can be written in *matrix form* as follows. If we set

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

then the system becomes

$$A\mathbf{x} = \mathbf{b}.$$

A solution of the system is a vector $\mathbf{v}$ of $\mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{b}$.

The $m \times n$ matrix A of all the coefficients is called the *coefficient matrix*, while $\mathbf{x}$ and $\mathbf{b}$ are, respectively, the vector of unknowns and the vector of constant terms. We will also consider the *augmented matrix*

$$C = [A|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \dots & \vdots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right].$$

Sometimes A is referred to as the *incomplete matrix*, whereas C is called *complete*.

It is also possible to represent a linear system $A\mathbf{x} = \mathbf{b}$ in a *vector form* as

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b},$$

where $\mathbf{c}_j$ denotes the j-th column of A, for $j = 1, \dots, n$.

Example 4.2. The linear system

$$\begin{cases} 2x_1 - x_2 + 5x_3 + x_4 = 1 \\ -x_1 + x_2 - 3x_3 + 6x_4 = 0 \\ x_2 - x_3 + x_4 = -8 \end{cases}$$

can be written as

$$\begin{bmatrix} 2 & -1 & 5 & 1 \\ -1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -8 \end{bmatrix}.$$

The augmented matrix is

$$C = \left[\begin{array}{cccc|c} 2 & -1 & 5 & 1 & 1 \\ -1 & 1 & -3 & 6 & 0 \\ 0 & 1 & -1 & 1 & -8 \end{array}\right].$$

The vector form is

$$x_1 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -8 \end{pmatrix}.$$

The matrix form and the vector form make it easy to find necessary and sufficient condition for a system to have solutions.

Theorem 4.1. *Rouché-Capelli's Theorem A linear system*

$$A\mathbf{x} = \mathbf{b}$$

of m equations in n unknowns has at least one solution, whatever m and n, if and only if the vector $\mathbf{b}$ is a linear combination of the columns of A, namely $\mathbf{b} \in \mathcal{C}(A) \subseteq \mathbb{R}^m$. It is equivalent to saying that $\operatorname{rank}(A) = \operatorname{rank}(A|\mathbf{b})$*, that is the coefficient matrix and the augmented matrix of the system must have the same rank.*

Proof We write the system in vector form as

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b}.$$

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ be a solution of the system. This means

$$u_1\mathbf{c}_1 + u_2\mathbf{c}_2 + \cdots + u_n\mathbf{c}_n = \mathbf{b}$$

so that $\mathbf{b}$ is now written as a linear combination of the columns of A: adding vector $\mathbf{b}$ to the column space of A does not modify the dimension of $\mathcal{C}(A)$ namely the rank of A, thus

$$\operatorname{rank} A = \operatorname{rank}(A|\mathbf{b}).$$

Conversely, if vector $\mathbf{b}$ is a linear combination of the columns of A, there are n real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\alpha_1\mathbf{c}_1 + \alpha_2\mathbf{c}_2 + \cdots + \alpha_n\mathbf{c}_n = \mathbf{b},$$

so we have a vector $\mathbf{a} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ such that $A\mathbf{a} = \mathbf{b}$, this means that $\mathbf{a}$ is a solution of the linear system. ■

Notice that, since the augmented matrix $(A|\mathbf{b})$ is obtained from the coefficient matrix A by adding just one column, the rank of $(A|\mathbf{b})$ is either equal to $\operatorname{rank}(A)$ or to $\operatorname{rank}(A) + 1$. According to the previous theorem, the system is solvable in the first case and impossible in the second one.

Example 4.3. 1. The system

$$\begin{cases} 2x_1 - x_2 = 1 \\ -x_1 + x_2 = 0 \\ x_1 - 3x_2 = -8 \end{cases}$$

has coefficient matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 1 & -3 \end{bmatrix}$ with $\operatorname{rank} A = 2$. The augmented matrix $M = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & -3 & -8 \end{bmatrix}$ has $\det M = -6 \neq 0$, so $\operatorname{rank} M = 3$ and the system has no solutions.

2. Consider the system

$$\begin{bmatrix} 2 & -1 & -1 \\ -4 & 5 & 3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

The coefficient matrix is of full rank since its determinant is $-2 \neq 0$. Moreover the augmented matrix has rank at most 3 since it has only 3 rows. It follows that $\operatorname{rank}(A) = \operatorname{rank}(A|\mathbf{b}) = 3$ and so the system has at least one solution.

4.3 Cramer systems

Before analyzing how to find the solutions of a linear system in the general situation, we focus on the case of a system $A\mathbf{x} = \mathbf{b}$, such that A is a non-singular square matrix, that is $A \in GL_n(\mathbb{R})$. Such a system is called a *Cramer system*.

Theorem 4.2 (Cramer's Theorem)**.** *Let $A\mathbf{x} = \mathbf{b}$ be a system with n linear equations and n unknowns and let A be non singular, i.e. $\det A \neq 0$. Then the system admits the unique solution*

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Proof Since matrix A is non singular, it has an inverse A^{-1}; multiplying on the left by A^{-1} both sides of the matrix equation $A\mathbf{x} = \mathbf{b}$ we get

$$A^{-1}A\mathbf{x} = I_n\mathbf{x} = \mathbf{x} = A^{-1}\mathbf{b},$$

hence the vector $A^{-1}\mathbf{b}$ is a solution. From the fact that the inverse of a matrix, when it exists, it is unique, it follows that the system has only one solution. ■

Example 4.4. The system

$$\begin{cases} x - y = 7 \\ 2x - 3y = 19 \end{cases}$$

in matrix form becomes

$$\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 19 \end{bmatrix}.$$

Since $\det A = -1$, it is a Cramer system. The inverse of matrix A is

$$\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix},$$

so the solution of the linear system is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 19 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

Finding the inverse of a matrix, generally requires many computations. For Cramer systems there is also an explicit rule giving the solution.

Theorem 4.3 (Cramer's rule)**.** *Let $A\mathbf{x} = \mathbf{b}$ be a Cramer system in n unknowns, i.e. A is an order n non singular square matrix. Denote with B_j the matrix obtained from A by replacing the j-th column with the vector $\mathbf{b}$, for $j = 1, \dots, n$. Then the solution is $A^{-1}\mathbf{b} = (u_1, \dots, u_n)$ where*

$$u_j = (\det A)^{-1} \det B_j \qquad \text{for } j = 1, \dots, n.$$

Example 4.5. The linear system

$$\begin{cases} x + y - z = 2 \\ x - y + z = 3 \\ -x + y + z = 4 \end{cases}$$

has coefficient matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ which is non-singular, since $\det A = -4$, therefore the linear system has the unique solution

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = A^{-1}\mathbf{b}.$$

To find the components u_1, u_2, u_3 of the solution we can apply Cramer's rule. So, we must calculate the determinants

$$|B_1| = \begin{vmatrix} 2 & 1 & -1 \\ 3 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix} = -10, \; |B_2| = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ -1 & 4 & 1 \end{vmatrix} = -12, \; |B_3| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \\ -1 & 1 & 4 \end{vmatrix} = -14.$$

Thus $u_1 = \dfrac{-10}{-4} = \dfrac{5}{2}, u_2 = \dfrac{-12}{-4} = 3, u_3 = \dfrac{-14}{-4} = \dfrac{7}{2}$, and the solution is $\left(\dfrac{5}{2}, 3, \dfrac{7}{2}\right)$.

4.4 The general case: Gauss method

The idea of Gauss method (also called elimination method) is to reduce the augmented matrix to an echelon form and then use "back substitution". This procedure relies on the following statement whose proof is straightforward.

Proposition 4.1. *Let C and C' be row equivalent matrices, that is, it is possible to pass from one to the other via elementary row operations. Then the two systems having C and C' as augmented matrices are equivalent, i.e. they have the same set of solutions.*

Notice that the same statement does not hold for columns equivalence.

The importance of the previous statement is due to the fact that if the augmented matrix is in row echelon form, the corresponding system is easy to solve. We say that a system is in *echelon form* if so it is its augmented matrix.

Example 4.6. 1. Consider the linear system in the unknowns x, y, z, t

$$\begin{cases} x - y + 2z - t = 1 \\ y + 2z + 6t = 0 \\ z + 4t = 2 \\ 0 = 3 \end{cases},$$

the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right).$$

It is in (row) echelon form and it is immediate to see that the system is impossible.

2. The linear system

$$\begin{cases} x - y + 2z = 1 \\ y + 2z = 0 \\ 3z = 9 \end{cases},$$

is in echelon form because so it is its augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & 9 \end{array}\right).$$

The last equation contains only the z unknown and we can solve it obtaining $z = 3$. By replacing z with 3 in the previous equation we get $y = -6$. Lastly, by replacing both z and y with the values we have found we obtain $x = -11$. So the only solution of the system is $(-11, -6, 3)$ and we have found it by a "back substitution" procedure.

3. The linear system

$$\begin{cases} x - y + 2z - t = 1 \\ 2z + 6t = 0 \\ 4t = 2 \end{cases},$$

has

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 4 & 2 \end{array}\right)$$

as augmented matrix. Proceeding as in the previous example, we find that $t = \frac{1}{2}$ and $z = -\frac{3}{2}$. Passing from the second to the first equation, two more unknowns are added: this means that we can choose one of them, let's say y, as a "parameter" and find the other. If we set $y = \alpha$, we get $x = y - 2z + t + 1 = \alpha + \frac{9}{2}$. So the general solution of the system is $\left(\alpha + \frac{9}{2}, \alpha, -\frac{3}{2}, \frac{1}{2}\right)$, i.e the set of (all) solutions is

$$S = \left\{ \left(\alpha + \frac{9}{2}, \alpha, -\frac{3}{2}, \frac{1}{2}\right) \mid \alpha \in \mathbb{R} \right\}.$$

In this case the system has infinite solutions since the free variable can be given any values whatsoever. If we want to find one of them, it is enough to choose a value for α. For example $\left(\frac{9}{2}, 0, -\frac{3}{2}, \frac{1}{2}\right)$ is the solution corresponding to $\alpha = 0$.

In the last example we choose as parameter the unknown whose corresponding column does not contain a pivot. If a system is in echelon form we call *pivots unknown or pivot variables* the ones corresponding to the columns of the augmented matrix containing the pivots and *free unknown or free variables or parameters* the others.

Summing up all the previous examples, we can use the following algorithm to solve a linear system

- write down the augmented matrix and, using elementary row operations, transform it to row echelon form: if the pivot of the last non-zero row lies in the last column (i.e in the one corresponding to constant terms) the system is impossible; otherwise
- erase the zero rows and rename the free variables, if there are any, then
- rewrite the corresponding system: since each row of the echelon form contains only a pivot, then each equation contains only a pivot variable. Use back substitution (from bottom to top) to find pivot variables.

It is clear that if, during the row reduction procedure, a row having only a non-zero element in the last column appears, then the system is impossible and there is no need to go ahead.

By Theorem 4.1 and recalling that the number of pivots in a equivalent echelon form is exactly the rank of a matrix, it follows that a real linear system $A\mathbf{x} = \mathbf{b}$, with $A \in \mathbb{R}^{m \times n}$, has

- no solution if and only if $\text{rank}(A|\mathbf{b}) > \text{rank}(A)$
- one solution if and only if $\text{rank}(A|\mathbf{b}) = \text{rank}(A) = n$, that is the rank equals the number of unknowns

- infinite solutions if and only if $\text{rank}(A|\mathbf{b}) = \text{rank}(A) < n$, that is the rank is less than the number of unknowns. In this case the solutions depend on $n - \text{rank}(A)$ parameters, each of which varies on the set of real numbers, so we say that the system has $\infty^{n-\text{rank}(A)}$ solutions.

As a consequence it is easy to see that a linear system with more unknowns than equations has either none or infinite solutions.

Remark

Consider a solvable linear system $A\mathbf{x} = \mathbf{b}$ and let r be the rank of A. By Theorem 3.5, it is possible to find a minor M in A of order r which is non-singular. Since $\text{rank}(A) = \text{rank}(A|\mathbf{b})$, the rows of the augmented matrix belonging to M are a basis for the row space of the augmented matrix. In terms of equations, this means that the equations corresponding to rows which are not in M are "linear combinations" of the others. It is now easy to see that this implies that the linear system $A\mathbf{x} = \mathbf{b}$ is equivalent to the one obtained by removing the equations corresponding to rows that are not in M.

We say that a linear system is *minimal* if the rank of the augmented matrix equals the number of equations. The above remark shows that each solvable system is equivalent to a minimal one.

4.5 Linear systems and subspaces of $\mathbb{R}^n$

A linear system with all constant terms equal to zero, namely $A\mathbf{x} = \mathbf{0}$, is said to be *homogeneous*. A homogeneous linear system is always solvable since, as it is evident, the null vector $\mathbf{0}$ is always a solution; it is described as the *trivial* solution.

A general result is the following.

Proposition 4.2. *The set* $\mathbf{S}$ *of all the solutions of a linear system* $A\mathbf{x} = \mathbf{b}$*, with* $A \in \mathbb{R}^{m\times n}$*, is a subspace of* $\mathbb{R}^n$ *if and only if* $\mathbf{b} = \mathbf{0}$*.*

Proof Consider a homogeneous system and let $S \subset \mathbb{R}^n$ be the set of solutions. We have to prove that $\mathbf{S}$ is closed under sum and scalar multiplication. If $\mathbf{z}_1$ and $\mathbf{z}_2$ are two solutions, i.e. $A\mathbf{z}_1 = \mathbf{0}$ and $A\mathbf{z}_2 = \mathbf{0}$, then

$$A(\mathbf{z}_1 + \mathbf{z}_2) = A\mathbf{z}_1 + A\mathbf{z}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

so $\mathbf{z}_1 + \mathbf{z}_2$ is a solution. Similarly, if $\mathbf{z}$ is a solution and $\alpha \in \mathbb{R}$

$$A(\alpha\mathbf{z}) = \alpha\,(A\mathbf{z}) = \alpha\mathbf{0} = \mathbf{0}.$$

so $\alpha\mathbf{z}$ is a solution. It follows that $\mathbf{S}$ is a subspace.

To prove the reverse statement, we suppose that the set $\mathbf{S}$ of solutions of a linear system $A\mathbf{x} = \mathbf{b}$ is a subspace of $\mathbb{R}^n$. Then the zero vector $\mathbf{0}$ of $\mathbb{R}^n$ belongs to $\mathbf{S}$ and so, by definition,

$$A\mathbf{0} = \mathbf{b}.$$

Since $A\mathbf{0} = \mathbf{0}$ it follows that $\mathbf{b} = \mathbf{0}$ and so the system is homogeneous. ■

So, if a linear system in n unknowns is homogeneous the set of solutions $\mathbf{S} = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$: it is natural to ask about its dimension.

Proposition 4.3. *Let $A \in \mathbb{R}^{m\times n}$ and set* $\mathrm{rank}A = r$. *The dimension of the subspace $\mathbf{S}$ of the solutions of the linear system $A\mathbf{x} = \mathbf{0}$ in n unknowns is*

$$\dim \mathbf{S} = n - r.$$

Proof If $r = n$ the system has only a solution which is the trivial one and $\dim(\{\mathbf{0}\}) = 0$.

If $r < n$, using the Gauss method, we can transform the system into an equivalent one which is in echelon form. The general solution of both the systems, depends on $k = n - r$ free variables and, without loss of generality, we can suppose that they are the last k. If we rename them as $\alpha_1, \ldots, \alpha_k$, the general solution of the system has the form

$$\mathbf{v} = \begin{pmatrix} d_{11}\alpha_1 + d_{12}\alpha_2 + \cdots + d_{1k}\alpha_k \\ d_{21}\alpha_1 + d_{22}\alpha_2 + \cdots + d_{2k}\alpha_k \\ \vdots \\ d_{r1}\alpha_1 + d_{r2}\alpha_2 + \cdots + d_{rk}\alpha_k \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \alpha_1 \begin{pmatrix} d_{11} \\ d_{21} \\ \vdots \\ d_{r1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} d_{12} \\ d_{22} \\ \vdots \\ d_{r2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \alpha_k \begin{pmatrix} d_{1k} \\ d_{2k} \\ \vdots \\ d_{rk} \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

So, if we set $\mathbf{d}_j = \begin{pmatrix} d_{1j} \\ d_{2j} \\ \vdots \\ d_{rj} \\ 0 \\ \vdots 1 \\ \\ \vdots \\ 0 \end{pmatrix}$, for $j = 1, \ldots, k$ then $\mathbf{S} = \mathbf{span}(\mathbf{d}_1, \ldots, \mathbf{d}_k)$. Moreover $\mathbf{d}_1, \ldots, \mathbf{d}_k$ are linearly independent: if C denotes the matrix having the $\mathbf{c}_j$'s as columns then C has rank k since it contains I_k as a minor. Summing up, $\{\mathbf{c}_1, \ldots, \mathbf{c}_k\}$ is a basis for $\mathbf{S}$ and so $\dim \mathbf{S} = k = n - r$. ■

Notice that the proof of the previous statement gives us a way to find a basis for the subspace of the solution of a homogeneous system: the vector $\mathbf{d}_j$ is the one obtained by choosing the value 1 for the i-th parameter and 0 for the others, for $j = 1, \ldots, n - r$. We call such solutions, *special* solutions of the linear system. Any other solution has the form

$$\mathbf{v} = \alpha_1\mathbf{d}_1 + \cdots + \alpha_{n-r}, \mathbf{d}_{n-r} \qquad \text{with } \alpha_1, \ldots, \alpha_{n-r} \in \mathbb{R},$$

that is it is linear combination of the special ones.

Example 4.7. The number of unknowns of the linear system

$$\begin{cases} x_1 - x_2 + 2x_4 + x_5 = 0 \\ x_2 + x_3 - x_4 - x_5 = 0 \end{cases}$$

is $n = 5$, the coefficient matrix is $A = \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 & -1 \end{bmatrix}$ that has rank $r = 2$, since, for example, the second row is not a multiple of the first. According to the previous result the dimension of the space of solutions S must be $n-r = 3$. The system is already in echelon form: the pivot variables are x_1 and x_2 and the free ones are x_3, x_4, x_5. By back substitution we get

$$\begin{cases} x_1 = -\alpha_1 - \alpha_2 \\ x_2 = -\alpha_1 + \alpha_2 + \alpha_3 \\ x_3 = \alpha_1 \\ x_4 = \alpha_2 \\ x_5 = \alpha_3 \end{cases}, \qquad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R},$$

that is, every solution is a vector of the form $\mathbf{v} = (-\alpha_1 - \alpha_2, -\alpha_1 + \alpha_2 + \alpha_3, \alpha_1, \alpha_2, \alpha_3)$. The special solutions are $\mathbf{v}_1 = (-1, -1, 1, 0, 0)$, for $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = 0$, $\mathbf{v}_2 = (-1, 1, 0, 1, 0)$, for $\alpha_2 = 1$ and $\alpha_1 = \alpha_3 = 0$, $\mathbf{v}_3 = (0, 1, 0, 0, 1)$, for $\alpha_3 = 1$ and $\alpha_1 = \alpha_2 = 0$ and we have

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3.$$

So a basis for S is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Of course the previous one is not the only choice for a basis of S: it is enough to find 3 linearly independent solutions. Another possible choice is $\{(1, 1, -1, 0, 0), (-2, 0, 1, 1, 0), (-1, 0, 1, 0, 1)\}$.

Summing up the two previous statements, the set of solution of a homogeneous linear system is subspace of $\mathbb{R}^n$ of dimension $n-r$, where n is the number of unknowns and r is the rank of the coefficients matrix. A converse statement is also true.

Proposition 4.4. *Let $W \subset \mathbb{R}^n$ be a proper subspace of dimension k. Then it is possible to find a minimal homogeneous linear system (with $n - k$ equations) whose set of solution is W.*

Proof Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be a basis for W. This means that $W = \mathbf{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ and so a vector $\mathbf{v} = (x_1, \ldots, x_n)$ of $\mathbb{R}^n$ belongs to W if and only if it is linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_k$. We can rephrase it by saying that $\mathbf{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k) = \mathbf{span}(\mathbf{v}, \mathbf{w}_1, \ldots, \mathbf{w}_k)$. Consider the $n \times (k+1)$ matrix A having the vectors as columns

$$A = (\mathbf{v}, \mathbf{w}_1, \ldots, \mathbf{w}_k).$$

It clearly has rank k and the last k columns are linearly independent: so there exists a non singular minor of order k not containing the first column. By

Kronecker's Theorem the minors of order $(k+1)$ constructed by adding to M the first column and one of the rows of A non included in M are singular: denote them with $M_1, \ldots, M_{n-k}$. So, the solutions of the linear system

$$\begin{cases} \det M_1 = 0 \\ \det M_2 = 0 \\ \ldots \\ \det M_{n-k} = 0 \end{cases}$$

are exactly the vectors of W. Moreover, the rank of the coefficient matrix is $n - \dim W = n - k$, that is the system is minimal. ■

It is possible to describe each proper subspace of $\mathbb{R}^n$ as the set of solutions of an homogeneous system: such representation is called Cartesian representation of the subspace. It is clear that a Cartesian representation is not unique: any linear system equivalent to a given Cartesian representation is itself a Cartesian representation. We can think of the equation $0 = 0$, as well as any other system equivalent to it, as a Cartesian, but not very useful, representation of $\mathbb{R}^n$ itself.

By Proposition 1.5, each element $\mathbf{x} = (x_1, \ldots, x_n)$ of a subspace W of $\mathbb{R}^n$ can be written as

$$\mathbf{x} = \alpha_1 \mathbf{w}_1 + \cdots + \alpha \mathbf{w}_h \qquad \text{with } \alpha_1, \ldots, \alpha_{n-r} \in \mathbb{R},$$

where $\mathbf{w}_1, \ldots, \mathbf{w}_h$ are generators of W. If we look at the components of this vector equation, we can rewrite it as

$$\begin{cases} x_1 = \alpha_1 d_{11} + \alpha_2 d_{12} + \cdots \alpha_h d_{1h} \\ x_2 = \alpha_1 d_{21} + \alpha_2 d_{22} + \cdots \alpha_h d_{2h} \\ \vdots \\ x_n = \alpha_1 d_{n1} + \alpha_2 d_{n2} + \cdots \alpha_h d_{nh} \end{cases},$$

Where $\mathbf{w}_j = (d_{1j}, \ldots, d_{nj})$, for $j = 1, \ldots, h$. An equivalent (matrix) form is

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = D \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_h \end{pmatrix},$$

where D denotes the matrix having the $\mathbf{w}_j$'s as columns. Such forms (both the "system" or the matrix one) are called parametric representations of the subspace W, since the components of the general element of the subspace are expressed in function of some parameters $\alpha_1, \ldots, \alpha_h$. Also parametric representation is clearly not unique but for all of them we have $\operatorname{rank}(D) = \dim W$. We say that a parametric representation is *minimal* if the number of parameters is equal to the dimension: equivalently, the columns of D are a basis for the subspace.

Example 4.8. Consider $W = \mathbf{span}((2,-2,1,0),(1,0,1,1),(-3,2,-1,-1)$. A parametric representation for W is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_n \end{pmatrix} = \begin{pmatrix} 2 & 1 & -3 \\ -2 & 0 & 2 \\ 1 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

Such representation is not minimal since

$$(-3,2,-2,-1) = -(2,-2,1,0) - (1,0,1,1).$$

A minimal representation is, for example

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_n \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

The dimension of W is 2 and the columns of the previous matrix form a basis. To find a Cartesian representation for W we observe that $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0$ and so write

$$\operatorname{rank} \begin{pmatrix} x_1 & 2 & 1 \\ x_2 & -2 & 0 \\ x_3 & 1 & 1 \\ x_4 & 0 & 1 \end{pmatrix} = 2 \Longleftrightarrow \begin{cases} \begin{vmatrix} x_1 & 2 & 1 \\ x_3 & 1 & 1 \\ x_4 & 0 & 1 \end{vmatrix} = 0 \\ \\ \begin{vmatrix} x_2 & -2 & 0 \\ x_3 & 1 & 1 \\ x_4 & 0 & 1 \end{vmatrix} = 0 \end{cases}.$$

By expanding the determinants, we get the minimal Cartesian representation

$$\begin{cases} x_1 - 2x_3 + x_4 = 0 \\ x_2 + 2x_3 - 2x_4 = 0 \end{cases}.$$

Now we come back to the general case: let $A\mathbf{x} = \mathbf{b}$ be a linear system, we say that $A\mathbf{x} = \mathbf{0}$ is the *associated* homogeneous system. The following result shows the relation between a solvable linear system and its associated homogeneous system.

Theorem 4.4. *Let $A\mathbf{x} = \mathbf{b}$ be a solvable linear system, with $A \in \mathbb{R}^{m\times n}$ and let $\mathbf{u}_0$ be a particular solution of it. Denote with S the set of all the solutions of the system and with S_0 the set of all the solutions of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$. Then*

$$S = \{\mathbf{u}_0 + \mathbf{v} \mid \mathbf{v} \in S_0\},$$

that is, each solution of S can be written as the sum of $\mathbf{u}_0$ and a solution of S_0.

Proof Suppose that $\mathbf{u}_0$ is a particular solution of the system $A\mathbf{x} = \mathbf{b}$, namely $A\mathbf{u}_0 = \mathbf{b}$, and $\mathbf{v}$ is any solution of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$, namely $A\mathbf{v} = \mathbf{0}$. Then $A(\mathbf{u}_0 + \mathbf{v}) = A\mathbf{u}_0 + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}$. Thus $\mathbf{u}_0 + \mathbf{v}$ too is a solution of the system $A\mathbf{x} = \mathbf{b}$.

Conversely, let $\mathbf{u}_0$ be a particular solution and $\mathbf{w}$ another solution of the system $A\mathbf{x} = \mathbf{b}$, namely $A\mathbf{u}_0 = \mathbf{b}$ and $A\mathbf{w} = \mathbf{b}$. Then $A(\mathbf{w}-\mathbf{u}_0) = A\mathbf{w}-A\mathbf{u}_0 = \mathbf{b}-\mathbf{b} = \mathbf{0}$. Thus $\mathbf{w} = \mathbf{u}_0+(\mathbf{w}-\mathbf{u}_0)$ where $\mathbf{w}-\mathbf{u}_0$ is a solution of the homogeneous system. ■

We can use the previous result to understand which is the structure of the set of the solutions of a solvable linear system. In Chapter 1, we described a geometrical vector also as an "action" shifting points in space. More generally, given a vector $\mathbf{a} \in \mathbb{R}^n$ we can define the *translation* associated to it as the bijection

$$t_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}^n$$

given by

$$t_{\mathbf{a}}(\mathbf{v}) = \mathbf{a} + \mathbf{v}.$$

If W is a subspace of $\mathbb{R}^n$ we can look at the subset $t_{\mathbf{a}}(W) = \{\mathbf{a} + \mathbf{w} \mid \mathbf{w} \in W\}$: unless $\mathbf{a} \in W$, the set $t_{\mathbf{a}}(W)$ is not a subspace of $\mathbb{R}^n$. Nevertheless, from the geometrical point of view, we can look at it as a "parallel" copy of W. Such a subset of $\mathbb{R}^n$ is called an *affine* subspace. More precisely, we have the following definition.

Definition 4.1. A subset L of $\mathbb{R}^n$ is an *affine subspace* if there exist a vector $\mathbf{a} \in \mathbb{R}^n$ and a vector subspace $W \subset \mathbb{R}^n$ such that $L = t_{\mathbf{a}}(W)$. If this is the case, we say that W is the *direction* of L.

So Theorem 4.4 can be rephrased by saying that $S = t_{\mathbf{u}_0}(S_0)$, that is the set S of the solutions of a linear system in n unknowns is an affine subspace of $\mathbb{R}^n$ having S_0 as direction.

4.6 The four fundamental subspaces of a matrix

In this section we keep the column notation for vectors in $\mathbb{R}^n$. Let $A \in \mathbb{R}^{m\times n}$, we can associate to A four *fundamental subspaces*

- the row space $\mathcal{R}(A)$, that is the subspace of $\mathbb{R}^n$ spanned by the rows of A,
- the column space $\mathcal{C}(A)$, that is the subspace of $\mathbb{R}^m$ spanned by the columns of A,
- the *nullspace* $\mathcal{N}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$, that is the subspace of $\mathbb{R}^n$ consisting of the solution of the homogeneous system having A as coefficient matrix,
- the *left nullspace* $\mathcal{N}(A^T) = \{\mathbf{y} \in \mathbb{R}^m \mid A^T\mathbf{y} = \mathbf{0}\}$, that is the subspace of $\mathbb{R}^m$ consisting of the solution of the homogeneous system having A^T as coefficient matrix.

Notice that $\mathcal{R}(A) = \mathcal{C}(A^T)$ and $\mathcal{C}(A) = \mathcal{R}(A^T)$. Moreover, the space $\mathcal{N}(A^T)$ coincides with the set $\{\mathbf{y} \in \mathbb{R}^m \mid y^T A = \mathbf{0}^T\}$. In fact

$$y^T A = \mathbf{0}^T \Leftrightarrow (y^T A)^T = (\mathbf{0}^T)^T \Leftrightarrow A^T \mathbf{y} = \mathbf{0}.$$

A similar computation shows that $\mathcal{N}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^T A^T = \mathbf{0}^T\}$.

From the results of the previous sections we can express the dimensions of the four subspaces in terms of the rank of A.

Proposition 4.5. *Let $A \in \mathbb{R}^{m\times n}$. Then*

- $\dim(\mathcal{R}(A)) = \dim(\mathcal{C}(A)) = \operatorname{rank}(A)$,
- $\dim(\mathcal{N}(A)) = n - \operatorname{rank}(A)$,
- $\dim(\mathcal{N}(A^T)) = m - \operatorname{rank}(A)$.

Example 4.9. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 2 & -1 & 2 & 1 \\ 1 & -2 & -2 & 2 \\ 3 & -3 & 0 & 3 \end{bmatrix}.$$

The null space and the row space lie in $\mathbb{R}^4$, while the column space and the left nullspace lie in $\mathbb{R}^5$. Consider the minor

$$M = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

it is non singular since $\det M = 1$ so $\operatorname{rank}(A) \geq 2$. Since $\mathbf{r}_5 = 3\mathbf{r}_1$, $\mathbf{r}_4 = \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r}_3 = 2\mathbf{r}_1 + \mathbf{r}_2$, the rank of A is 2 and the minor M realizes the rank. Then $\dim(\mathcal{R}(A)) = \dim(\mathcal{C}(A)) = 2$, $\dim(\mathcal{N}(A)) = 4 - 2 = 2$ and $\dim(\mathcal{N}(A^T)) = 5 - 2 = 3$. Moreover the first two rows are a basis of $\mathcal{R}(A)$ and the general element of the row space has the form

$$\mathbf{v} = \alpha(1, -1, 0, 1) + \beta(0, 1, 2, -1), \qquad \alpha, \beta \in \mathbb{R}.$$

The linear system $A\mathbf{x} = \mathbf{0}$ is the equivalent to

$$\begin{cases} x_1 - x_2 + x_4 = 0 \\ x_2 + 2x_3 - x_4 = 0 \end{cases},$$

and gives a minimal Cartesian representation for the nullspace.
Similarly the first two columns of A form a basis of the column space, that is $\mathcal{C}(A) = \mathbf{span}((1, 0, 2, 1, 3), (-1, 1, -1, -2, -3))$ and the left nullspace is represented by the following minimal system

$$(y_1, y_2, y_3, y_4, y_5) \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \\ 1 & -2 \\ 3 & -3 \end{bmatrix} = (0, 0) \qquad \Rightarrow \qquad \begin{cases} y_1 + 2y_3 + y_4 + 3y_5 = 0 \\ -y_1 + y_2 - y_3 - 2y_4 - 3y_5 = 0 \end{cases}.$$

Chapter 5

5 The Euclidean structure of $\mathbb{R}^n$

5.1 The inner (dot) product

In this chapter we deal with the structure induced on $\mathbb{R}^n$ by inner product. We introduce it in Chapter 2 in order to define row-by-column product: let's recall its definition.

Definition 5.1. In the vector space $\mathbb{R}^n$ the *inner product* or *dot product* (sometimes *scalar product)* of two vectors $\mathbf{u} = (x_1, x_2, \ldots, x_n),\ \mathbf{v} = (y_1, y_2, \ldots, y_n)$ is a real number denoted as $\langle \mathbf{u}, \mathbf{v} \rangle$, or sometimes $\mathbf{u} \cdot \mathbf{v}$, given by

$$\langle \mathbf{u}, \mathbf{v} \rangle = x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

For example, if $\mathbf{u} = (0, -1, 0, 1), \mathbf{v} = (1, 4, 0, 1) \in \mathbb{R}^4$ then

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0 \cdot 1 + (-1) \cdot 4 + 0 \cdot 0 + 1 \cdot 1 = -3.$$

It is easy to see from the definition that the inner product in $\mathbb{R}^n$ has the following properties

1. *Symmetry*
$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$
for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

2. *Linearity*
$$\langle \alpha \mathbf{u}_1 + \beta \mathbf{u}_2, \mathbf{v} \rangle \quad = \quad \alpha \langle \mathbf{u}_1, \mathbf{v} \rangle + \beta \langle \mathbf{u}_2, \mathbf{v} \rangle$$
for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v} \in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{R}$

3. *Positivity*
$$\langle \mathbf{u}, \mathbf{u} \rangle = \sum_{i=1}^{n} x_i^2 \geq 0 \text{ and } \langle \mathbf{u}, \mathbf{u} \rangle = 0 \iff \mathbf{u} = \mathbf{0}$$
for all $\mathbf{u} \in \mathbb{R}^n$.

Note that for a vector $\mathbf{u} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$,

$$\langle \mathbf{u}, \mathbf{e}_i \rangle = x_i \ \forall i = 1, ..., n$$

where $\mathbf{e}_i = (0, 0, ..., 1, ...0)$ is the $i-$th element of the standard basis.

By means of $\langle\ ,\ \rangle$, some elementary geometry concepts like length, distance between two points, orthogonality and angle between two vectors can be extended to any $\mathbb{R}^n$.

5.2 Euclidean norm

We define Euclidean *norm* or *length* of a vector $\mathbf{u} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ the non negative number

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

For example if $\mathbf{u} = (1, 0, -2, 5) \in \mathbb{R}^4$, then $\|\mathbf{u}\| = \sqrt{1^2 + 0^2 + (-2)^2 + 5^2} = \sqrt{30}$.

For $n = 1, 2, 3$ we find the already well-known formulas from Euclidean geometry on the line, in the plane and in the 3-dimensional space

$$\|\mathbf{u}\| = |x_1|, \quad \|\mathbf{u}\| = \sqrt{x_1^2 + x_2^2}, \quad \|\mathbf{u}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Proposition 5.1. *The following properties hold true*

1. $\|\mathbf{u}\| \geq 0$ *and* $\|\mathbf{u}\| = 0 \iff \mathbf{u} = \mathbf{0}$
2. $\|\alpha \mathbf{u}\| = |\alpha| \|\mathbf{u}\|$. *So the length of a vector and of its opposite are the same.*
3. *Triangular inequality*
$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$
4. *Cauchy-Schwarz inequality (C-S inequality)*
$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *and for all* $\alpha \in \mathbb{R}$.

Proof The first two properties follow directly from the properties of the scalar product.

The C-S inequality can be proved from the obvious statement

$$\langle \mathbf{u} + \lambda \mathbf{v}, \mathbf{u} + \lambda \mathbf{v} \rangle \geq 0 \quad \text{for all} \quad \lambda \in \mathbb{R}$$

which implies

$$\langle \mathbf{u}, \mathbf{u} \rangle + 2\lambda \langle \mathbf{u}, \mathbf{v} \rangle + \lambda^2 \langle \mathbf{v}, \mathbf{v} \rangle \geq 0,$$

for all $\lambda \in \mathbb{R}$. The left-hand side is a second degree polynomial in the variable λ, which is never negative. Hence the second degree equation

$$\langle \mathbf{u}, \mathbf{u} \rangle + 2\lambda \langle \mathbf{u}, \mathbf{v} \rangle + \lambda^2 \langle \mathbf{v}, \mathbf{v} \rangle = 0$$

does **not** have two real solutions, so its discriminant must be less than or equal to zero

$$\Delta = 4\langle \mathbf{u}, \mathbf{v}\rangle^2 - 4\langle \mathbf{u}, \mathbf{u}\rangle \ \langle \mathbf{v}, \mathbf{v}\rangle \leq 0.$$

Thus

$$\langle \mathbf{u}, \mathbf{v}\rangle^2 \leq \langle \mathbf{u}, \mathbf{u}\rangle \ \langle \mathbf{v}, \mathbf{v}\rangle$$

and so

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle \langle \mathbf{v}, \mathbf{v}\rangle} = ||\mathbf{u}|| \ ||\mathbf{v}||.$$

The triangular inequality is equivalent to its square

$$\|\mathbf{u} + \mathbf{v}\|^2 \leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^2.$$

We have

$$\|\mathbf{u} + \mathbf{v}\|^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}\rangle = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2,$$

and

$$(\|\mathbf{u}\| + \|\mathbf{v}\|)^2 = \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2.$$

The triangular inequality is thus equivalent to

$$\langle \mathbf{u}, \mathbf{v}\rangle \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

which follows from C-S, since

$$\langle \mathbf{u}, \mathbf{v}\rangle \leq |\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|. \qquad \blacksquare$$

Given two non-null vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, from the Cauchy-Schwarz inequality

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

it follows that

$$\left| \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} \right| \leq 1$$

i.e.

$$-1 \leq \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1.$$

For this reason the number $\dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}$ is always the cosine of a suitable angle $\theta \in [0, \pi]$, namely

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|},$$

and $\dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}$ is called the *cosine of the angle between vectors* $\mathbf{u}$ and $\mathbf{v}$. We can also write

$$\theta = \arccos \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} \in [0, \pi],$$

and call θ the *angle* between $\mathbf{v}$ and $\mathbf{u}$.

For example if $\mathbf{u} = (-4, 2, 1, 2)$ and $\mathbf{v} = (1, 0, 2, 2)$, then $\cos\theta = \dfrac{2}{15}$ and $\theta \sim 1,437$ radians.

The notion of angle between two vectors in $\mathbb{R}^n$, coincides, for $n = 2, 3$, with the geometric notion of angle in the plane or in the 3-dimensional space and extends it to higher dimensional space.

A *unit* vector is a vector whose length is equal to 1. For instance all the vectors of the standard basis of $\mathbb{R}^n$ are unit vectors.

Every non zero vector $\mathbf{u}$ has two unit vectors proportional to it that are

$$\pm\frac{\mathbf{1}}{\|\mathbf{u}\|}\mathbf{u}$$

since, by the above properties,

$$\| \pm \frac{\mathbf{1}}{\|\mathbf{u}\|}\mathbf{u}\| = \left|\frac{\mathbf{1}}{\|\mathbf{u}\|}\right| \|\mathbf{u}\| = \frac{\mathbf{1}}{\|\mathbf{u}\|}\|\mathbf{u}\| = 1.$$

Replacing $\mathbf{u}$ by either of them is described as *normalizing* $\mathbf{u}$.

Example 5.1. The norm of $\mathbf{u} = (1, 1, 2, -1)$ is $\sqrt{7}$, so normalizing it means replacing it by either $\mathbf{u}^* = (\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{2}{\sqrt{7}}, -\frac{1}{\sqrt{7}})$ or $-\mathbf{u}^*$.

5.3 Orthogonal vectors

Two vectors $\mathbf{u} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{v} = (y_1, y_2, \ldots, y_n)$ are called *orthogonal* or *perpendicular* if

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

For instance, in $\mathbb{R}^4$ vectors $\mathbf{u} = (1, 0, 2, -1)$ and $\mathbf{v} = (3, 1, 0, 3)$ are orthogonal. It is easy to see that any two vectors of the standard basis of $\mathbb{R}^n$ are orthogonal.

The null vector $\mathbf{0}$ is orthogonal to all the vectors of $\mathbb{R}^n$

$$\langle \mathbf{u}, \mathbf{0} \rangle = 0$$

for all $\mathbf{u} \in \mathbb{R}^n$. Moreover, this property characterizes the zero vector: if a vector of $\mathbb{R}^n$ is orthogonal to all vectors of $\mathbb{R}^n$ then it is orthogonal also to itself and so it is the zero vector.

Proposition 5.2. *Given vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$*, the following statements are equivalent*

1. $\langle \mathbf{u}, \mathbf{v} \rangle = 0$
2. $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ (Pythagoras' theorem)
3. $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$

4. the angle between $\mathbf{u}$ *and* $\mathbf{v}$ *is* $\dfrac{\pi}{2}$.

Proof Note that $\|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|\|^2$, which gives the equivalence of 1. and 2. Similarly, from $\|\mathbf{u}-\mathbf{v}\|^2 = \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle = \|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2$ there follows the equivalence of 1. and 3. The equivalence of 1. and 4. follows from the definition of angle between two vectors. ■

Another important property of orthogonal vectors is the following.

Proposition 5.3. *Two non zero orthogonal vectors are linearly independent.*

Proof We prove the statement by reductio ad absurdum. Denote with $\mathbf{v}$ and $\mathbf{u}$ the two vectors and suppose that they are proportional. Then we have $\mathbf{u} = \alpha\mathbf{v}$ with $\alpha \in \mathbb{R}$ different from zero (since $\mathbf{u} \neq \mathbf{0}$). So

$$0 = \langle \mathbf{u}, \mathbf{v}\rangle = \langle \alpha\mathbf{v}, \mathbf{v}\rangle = \alpha\langle \mathbf{v}, \mathbf{v}\rangle = \alpha\|\mathbf{v}\| \;\Rightarrow\; \|\mathbf{v}\| = 0,$$

since $\alpha \neq 0$. But this is impossible because it would imply $\mathbf{v} = \mathbf{0}$. ■

5.4 Orthogonal complement

Let S and T be two subsets of $\mathbb{R}^n$. We say that S and T are *orthogonal* if every vector in S is orthogonal to every vector in T

$$\langle \mathbf{v}, \mathbf{w}\rangle = 0 \text{ for all } \mathbf{v} \in S \text{ and } \mathbf{w} \in T.$$

For example, in $\mathbb{R}^3$ the sets $S = \{(1,0,0), (0,2,0)\}$ and $T = \{(0,0,-1)\}$ are orthogonal.

The same definition apply also to subspace of $\mathbb{R}^n$, even if non null subspaces contain an infinite number of vectors. Fortunately, the next result shows that in order to understand whether two subspaces are orthogonal or not it is enough to check the orthogonality only of a finite number of vectors.

Proposition 5.4. *Two non null subspaces* W, U *of* $\mathbb{R}^n$ *are orthogonal if and only if any two finite sets of generators* S *and* T *for* W *and* U*, respectively, are orthogonal.*

Proof We have $S \subset W$ and $T \subset U$, so it is clear that the orthogonality of W and U implies the orthogonality of S and T. Conversely, suppose that $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ and $T = \{\mathbf{u}_1, \ldots, \mathbf{u}_h\}$ are orthogonal, namely $\langle \mathbf{w}_j, \mathbf{u}_i\rangle = 0$ for each $j = 1, \ldots, k$ and $i = 1, \ldots, h$. We have to prove that $\langle \mathbf{w}, \mathbf{u}\rangle = 0$ for each $\mathbf{w} \in W$ and each $\mathbf{u} \in U$. Since $W = \mathbf{span}(S)$ and $U = \mathbf{span}(T)$ we have

$$\mathbf{w} = \alpha_1\mathbf{w}_1 + \cdots + \alpha_k\mathbf{w}_k \qquad \mathbf{u} = \beta_1\mathbf{u}_1 + \cdots + \beta_h\mathbf{u}_h,$$

for some $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_h \in \mathbb{R}$. Using the properties of the dot product

$$\begin{aligned}\langle \mathbf{w}, \mathbf{u} \rangle &= \langle \sum_{j=1}^{k} \alpha_j \mathbf{w}_j, \mathbf{u} \rangle = \sum_{j=1}^{k} \alpha_j \langle \mathbf{w}_j, \mathbf{u} \rangle = \\ &= \sum_{j=1}^{k} \alpha_j \langle \mathbf{w}_j, \sum_{i=1}^{h} \beta_i \mathbf{u}_i \rangle = \sum_{j=1}^{k} \sum_{i=1}^{h} \alpha_j \beta_i \langle \mathbf{w}_j, \mathbf{u}_i \rangle = 0\end{aligned}$$

since $\mathbf{w}_j$ and $\mathbf{u}_i$ are orthogonal for each $j = 1, \ldots, k$ and $i = 1, \ldots, h$. ■

In $\mathbb{R}^3$ the xy-plane $W = \{(x, y, 0) \mid z = 0\}$ and the z-axis $U = \{(0, 0, a) \mid a \in \mathbb{R}\}$ are orthogonal since $S = \{(1, 0, 0), (0, 1, 0)\}$ and $T = \{(0, 0, 1)\}$ are orthogonal generating sets.

Notice that if a vector $\mathbf{v}$ is in two orthogonal subsets, it must be orthogonal to itself, that is $\langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|^2 = 0$. This has to be the zero vector. So any two orthogonal subspaces have only the zero vector in common.

Let $W \subset \mathbb{R}^n$ be a subspace. The *orthogonal complement* of W is the subset of V containing all the vectors of $\mathbb{R}^n$ orthogonal to W that is

$$W^{\perp} = \{\mathbf{u} \in \mathbb{R}^n \mid \langle \mathbf{u}, \mathbf{w} \rangle = 0, \text{for all } w \in W\}.$$

It is easy to check that $W^{\perp}$ is a subspace of $\mathbb{R}^n$ and, in fact, it is the maximal one, with respect to inclusion. Another important property we won't prove is that $(W^{\perp})^{\perp} = W$.

We want to compare the dimensions of W and $W^{\perp}$. By Theorem 1.3 we have

$$\dim(W^{\perp}) = \dim(W + W^{\perp}) + \dim(W \cap W^{\perp}) - \dim W \leq n - \dim W,$$

since $W \cap W^{\perp} = \{\mathbf{0}\}$ and $\dim(W + W^{\perp}) \leq n$. In fact the equality holds, before stating the result we look at some examples.

Example 5.2.

1. In $\mathbb{R}^5$ consider the subspace $W = \mathbf{span}\{(1, 1, 1, 1, 0), (1, 0, 0, 1, 0)\}$. A vector $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ is orthogonal to each vector of W if and only if $\langle (1, 1, 1, 1, 0), \mathbf{x} \rangle = 0$ and $\langle (1, 0, 0, 1, 0), \mathbf{x} \rangle = 0$. This means that

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0} \iff \begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_4 = 0 \end{cases}$$

   is a Cartesian representation for $W^{\perp}$. Note that $\dim W = 2$, since the two vectors generating it are linearly independent. So the Cartesian representation we get is minimal and $\dim(W^{\perp}) = 5 - 2 = 3$.

2. In $\mathbb{R}^3$ consider the subspace $U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0, x + y = 0\}$. It is a line since the two equation defining U are linearly independent. We can rewrite U as

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0} \iff \begin{cases} \langle (1, 1, 1), (x, y, z) \rangle = 0 \\ \langle (1, 1, 0), (x, y, z) \rangle = 0 \end{cases}$$

This means that $(1,1,1),(1,1,0) \in U^\perp$: in fact, since $\dim(U^\perp) \leq 3-1 = 2$, we have $\mathbf{span}((1,1,1),(1,1,0)) = U^\perp$.

Theorem 5.1. *Let W be a subspace of $\mathbb{R}^n$.*

1. *If $\mathbf{x} = D\underline{\alpha}$, with $\underline{\alpha}$ a vector of parameters, is a parametric representation for W then $D^T\mathbf{x} = \mathbf{0}$ is a Cartesian representation of $W^\perp$.*
2. *If $A\mathbf{x} = \mathbf{0}$, is a Cartesian representation for W then $\mathbf{x} = A^T\underline{\alpha}$, with $\underline{\alpha}$ a vector of parameters, is a parametric representation of $W^\perp$.*
3. *$\mathbb{R}^n = W \oplus W^\perp$. In particular, $\dim(W^\perp) = n - \dim W$.*

Moreover if we start with a minimal representation of W we get a minimal representation for $W^\perp$.

We won't demonstrate the theorem: however the proof is not very different from the above examples. If we apply the theorem to the four fundamental subspaces of a matrix we get the following result.

Proposition 5.5. *Let $A \in \mathbb{R}^{m\times n}$ then*

1. *$(\mathcal{R}(A))^\perp = \mathcal{N}(A)$, so $\mathbb{R}^n = \mathcal{R}(A) \oplus \mathcal{N}(A)$,*
2. *$(\mathcal{C}(A))^\perp = \mathcal{N}(A^T)$, so $\mathbb{R}^m = \mathcal{C}(A) \oplus \mathcal{N}(A^T)$.*

5.5 Euclidean distance

Let $\mathbf{u} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{v} = (y_1, y_2, \ldots, y_n)$ be two vectors of $\mathbb{R}^n$. The *Euclidean distance* from $\mathbf{u}$ to $\mathbf{v}$ is

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}.$$

It is an extension to any dimension n of the well-known distances for $n = 1, 2, 3$ between two points on the line, in the plane and in the space with respect to a given reference system.

For example if $\mathbf{u} = (1,-2,0,3,1), \mathbf{v} = (0,-3,1,-2,0) \in \mathbb{R}^4$, their Euclidean distance is

$$\|\mathbf{u}-\mathbf{v}\| = \sqrt{(1-0)^2 + (-2+3)^2 + (0-1)^2 + (3+2)^2 + (1-0)^2} = \sqrt{29}.$$

Note that the Euclidean distance from $\mathbf{u}$ to the zero vector $\mathbf{0}$ is precisely the Euclidean norm of $\mathbf{u}$

$$d(P,O) = d(\mathbf{u},\mathbf{0}) = \|\mathbf{u}-\mathbf{0}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

From the properties of the Euclidean distance it follows that, for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}^n$

1. $d(\mathbf{u},\mathbf{v}) \geq 0$ and $d(\mathbf{u},\mathbf{v}) = 0 \iff \mathbf{u} = \mathbf{v}$ (positivity)

2. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$ (symmetry)

3. $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ (triangular inequality).

Euclidean distance gives to $\mathbb{R}^n$ the structure of a "metric space" and, for example, allows to extend to $\mathbb{R}^n$ many concepts of analysis such as the one of limit.

Chapter 6

6 Orthogonal projections and least square approximations

6.1 Projection onto a 1-dimensional subspace

Let $\mathbf{b}$ be a non zero vector of $\mathbb{R}^n$ and denote with $\mathbf{S} = \mathbf{span}\ (\mathbf{b}) = \{\lambda\mathbf{b} \mid \lambda \in \mathbb{R}\}$ the 1-dimensional subspace spanned by $\mathbf{b}$, (that is the line through the origin having $\mathbf{b}$ as direction).
Given a vector $\mathbf{v} \in \mathbb{R}^n$ we want to find, if it is possible, the vector of $\mathbf{span}(\mathbf{b})$ nearest to $\mathbf{v}$.
We can rephrase the problem as follows: we want to minimize the function

$$f(\lambda) = d(\mathbf{v}, \lambda\mathbf{b}) = \|\mathbf{v} - \lambda\mathbf{b}\|,$$

for $\lambda \in \mathbb{R}$.

Remark Note that if λ^* minimizes a non-negative function $h(\lambda)$, then λ^* minimizes also $h^2(\lambda)$ since

$$0 \leq h(\lambda^*) \leq h(\lambda) \Longleftrightarrow 0 \leq h^2(\lambda^*) \leq h^2(\lambda).$$

This remark implies that minimizing $\|\mathbf{v} - \lambda\mathbf{b}\|$ is the same as minimizing $\|\mathbf{v} - \lambda\mathbf{b}\|^2$. We have

$$g(\lambda) = \|\mathbf{v} - \lambda\mathbf{b}\|^2 = \|\mathbf{v}\|^2 - 2\lambda\langle\mathbf{v}, \mathbf{b}\rangle + \lambda^2\|\mathbf{b}\|^2,$$

The derivative g' of g is

$$g'(\lambda) = -2\langle\mathbf{v}, \mathbf{b}\rangle + 2\lambda\|\mathbf{b}\|^2.$$

So it has

$$\lambda^* = \frac{\langle\mathbf{v}, \mathbf{b}\rangle}{\|\mathbf{b}\|^2} = \frac{\langle\mathbf{v}, \mathbf{b}\rangle}{\langle\mathbf{b}, \mathbf{b}\rangle}.$$

as critical point. Since

$$g''(\lambda^*) = 2\|\mathbf{b}\|^2 > 0,$$

λ^* is a point of minimum for $g(\lambda)$ and so the vector of $\mathbf{span}(\mathbf{b})$ realizing the minimum distance from $\mathbf{v}$ is

$$\boxed{\mathbf{v}_1 = \frac{\langle\mathbf{v}, \mathbf{b}\rangle}{\langle\mathbf{b}, \mathbf{b}\rangle}\mathbf{b}.}$$

The scalar $\dfrac{\langle\mathbf{v}, \mathbf{b}\rangle}{\langle\mathbf{b}, \mathbf{b}\rangle}$ is called *Fourier coefficient* of $\mathbf{v}$ with respect to $\mathbf{b}$.

Proposition 6.1. *The vector* $\mathbf{v} - \mathbf{v}_1 = \mathbf{v} - \dfrac{\langle \boldsymbol{v}, \mathbf{b}\rangle}{\langle \mathbf{b}, \mathbf{b}\rangle}\mathbf{b}$ *is orthogonal to* $\mathbf{b}$*, i.e.* $\mathbf{v} - \mathbf{v}_1 \in \mathbf{span}(\mathbf{b})^{\perp}$.

Proof From the linearity and symmetry properties of scalar product we have

$$\langle \mathbf{b}, \mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{b}\rangle}{\langle \mathbf{b}, \mathbf{b}\rangle}\mathbf{b}\rangle = \langle \mathbf{b}, \mathbf{v}\rangle - \langle \mathbf{b}, \frac{\langle \mathbf{v}, \mathbf{b}\rangle}{\langle \mathbf{b}, \mathbf{b}\rangle}\mathbf{b}\rangle = \langle \mathbf{b}, \mathbf{v}\rangle - \frac{\langle \mathbf{v}, \mathbf{b}\rangle}{\langle \mathbf{b}, \mathbf{b}\rangle}\langle \mathbf{b}, \mathbf{b}\rangle = 0. \qquad \blacksquare$$

By the above results, we have decomposed the vector $\mathbf{v}$ as

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$

with $\mathbf{v}_1$ proportional to $\mathbf{b}$ and $\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1$ orthogonal to $\mathbf{b}$.
Such a decomposition is *unique*. In fact, if

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_1^* + \mathbf{v}_2^*$$

with $\mathbf{v}_1$ e $\mathbf{v}_1^*$ proportional to $\mathbf{b}$, and $\mathbf{v}_2, \mathbf{v}_2^*$ orthogonal to $\mathbf{b}$, the vector

$$\mathbf{v}_1 - \mathbf{v}_1^* = \mathbf{v}_2^* - \mathbf{v}_2 \in \mathbf{span}(\mathbf{b}) \cap \mathbf{span}(\mathbf{b})^{\perp}$$

and so it is the zero vector, since $\mathbf{span}(\mathbf{b}) \cap \mathbf{span}(\mathbf{b})^{\perp}$ is always the null subspace.

The vector $\mathbf{v}_1$ is called *orthogonal projection* or simply the *projection* of $\mathbf{v}$ onto (the 1-dimensional subspace spanned by) $\mathbf{b}$ and is denoted with

$$\mathrm{p}_{\mathbf{b}}\mathbf{v}.$$

By definitions, it is the best approximation of $\mathbf{v}$ with a vector proportional to $\mathbf{b}$.

Example 6.1. 1. In $\mathbb{R}^4$, consider $\mathbf{b} = (1, 0, 2, 1)$. We want to find the projections of the vectors

- $\mathbf{v} = (0, 0, -3, 1)$
- $\mathbf{w} = (-1, 0, -2, -1)$
- $\mathbf{u} = (1, 1, 0, -1)$

onto $\mathbf{b}$.
Using the formula $\mathrm{p}_{\mathbf{b}}\mathbf{v} = \dfrac{\langle \mathbf{v}, \mathbf{b}\rangle}{\langle \mathbf{b}, \mathbf{b}\rangle}\mathbf{b}$, we get

- $\mathrm{p}_{\mathbf{b}}\mathbf{v} = -\frac{5}{6}(1, 0, 2, 1)$
- $\mathrm{p}_{\mathbf{b}}\mathbf{w} = (-1, 0. - 2, -1)$
- $\mathrm{p}_{\mathbf{b}}\mathbf{u} = (0, 0, 0, 0) = \mathbf{0}$.

2. In $\mathbb{R}^4$ we are looking for the best approximation of $\mathbf{a} = (6, -2, 5, 3)$ with a vector having all entries equal to each others.
   That is, we want to find a vector $(\alpha, \alpha, \alpha, \alpha) = \alpha(1, 1, 1, 1)$ having minimal distance from $\mathbf{a}$. So, the required vector is

$$\mathrm{p}_{(1,1,1,1)}\mathbf{a} = \frac{12}{4}(1, 1, 1, 1) = (3, 3, 3, 3)$$

   Note that 3 is the average of the components of $\mathbf{a}$.

3. It is not difficult to prove that

$$\mathrm{p}_{\mathbf{b}}\mathbf{v} = \mathbf{v} \iff \mathbf{v} \in \mathbf{span}(\mathbf{b}),$$

   and

$$\mathrm{p}_{\mathbf{b}}\mathbf{v} = \mathbf{0} \iff \langle \mathbf{v}, \mathbf{b} \rangle = 0 \iff \mathbf{v} \in \mathbf{span}(\mathbf{b})^{\perp}.$$

6.2 Orthogonal and orthonormal basis

Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r\} \subset \mathbb{R}^n$ be a finite set; we say that A is *orthogonal* if the vector of A are orthogonal to each others, that is

$$\langle \mathbf{a}_i, \mathbf{a}_j \rangle = 0, \quad i \neq j, \quad i, j = 1, \ldots, r.$$

Moreover, we say that A is *orthonormal* if A is orthogonal and each vector of A is a unit vector, i.e. $\|\mathbf{a_i}\| = \mathbf{1}$, for each $i = 1, \ldots, r$. Clearly each orthonormal set is also orthogonal, but the converse is not true as the set $\{(1, 1), (1, -1)\} \subset \mathbb{R}^2$ shows. Clearly the canonical basis of $\mathbb{R}^n$ is an orthonormal set.

The following statement generalizes Proposition 5.3.

Proposition 6.2. *If $A = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r\}$ is an orthogonal set with $\mathbf{a}_i \neq \mathbf{0}$ for each $i = 1, \ldots, r$, then it is linearly independent.*

As a consequence the maximum number of non zero orthogonal vectors in $\mathbb{R}^n$ is n.

We say that a basis of $\mathbb{R}^n$ is *orthogonal* (resp. *orthonormal*) if it is an orthogonal set (resp orthonormal set). As we already observed, the canonical basis of $\mathbb{R}^n$, is orthogonal (and also orthonormal), but it is not the only one: for example, the set $B = \{(1, 1, 0), (-1, 1, 2), (1, -1, 1)\}$ is an orthogonal basis of $\mathbb{R}^3$: it is an orthogonal set, consisting of 3 non zero vectors, and so it is a linearly independent set spanning $\mathbb{R}^3$. An example of an orthonormal basis of $\mathbb{R}^4$, different from the canonical one is,

$$H = \left\{ \frac{1}{\sqrt{3}}(1, -1, 0, 1), (0, 0, 1, 0), \frac{1}{\sqrt{2}}(0, 1, 0, 1), \frac{1}{\sqrt{6}}(-2, -1, 0, -1) \right\}.$$

Since the restriction of the inner product of $\mathbb{R}^n$ to a subspace $W \subset \mathbb{R}^n$ still satisfies all the properties of an inner product, each subspace W has itself the structure of an Euclidean space and so we can talk about orthogonal or orthonormal basis for a subspace. For example, in $\mathbb{R}^4$, the set

$$A = \{(1,-1,1,0),(2,2,0,1),(0,1,1,-2)\}$$

is an orthogonal basis for the subspace $W = \mathbf{span}(A)$ of dimension 3.

We end the section by observing that, given an orthogonal basis $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r\}$ for a subspace W, it is possible to find an orthonormal basis by normalizing each vector of A. For example the set

$$B = \left\{ \frac{1}{\sqrt{3}}(1,-1,1,0), \frac{1}{3}(2,2,0,1), \frac{1}{\sqrt{6}}(0,1,1,-2) \right\}$$

is an orthonormal basis of $W = \mathbf{span}(A)$.

6.3 Gram-Schmidt process

The aim of this section is to prove that each non null subspace of $\mathbb{R}^n$ admits an orthogonal (and, up to normalization, orthonormal) basis.

Theorem 6.1 (Gram-Schmidt process)**.** *Each non null subspace of* $\mathbb{R}^n$ *has an orthogonal basis.*

Proof Let $B = \{\mathbf{w}_1, \ldots, \mathbf{w}_r\}$ be a basis for a non null subspace W of $\mathbb{R}^n$. We construct the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ as follows

$$\begin{aligned}
&\mathbf{u}_1 = \mathbf{w}_1 \\
&\mathbf{u}_2 = \mathbf{w}_2 - \mathrm{p}_{\mathbf{u_1}}\mathbf{w_2} = \mathbf{w}_2 - \frac{\langle \mathbf{w_2}, \mathbf{u_1} \rangle}{\langle \mathbf{u_1}, \mathbf{u_1} \rangle}\mathbf{u_1} \\
&\mathbf{u}_3 = \mathbf{w}_3 - \mathrm{p}_{\mathbf{u_1}}\mathbf{w_3} - \mathrm{p}_{\mathbf{u_2}}\mathbf{w_3} = \mathbf{w}_3 - \frac{\langle \mathbf{w_3}, \mathbf{u_1} \rangle}{\langle \mathbf{u_1}, \mathbf{u_1} \rangle}\mathbf{u_1} - \frac{\langle \mathbf{w_3}, \mathbf{u_2} \rangle}{\langle \mathbf{u_2}, \mathbf{u_2} \rangle}\mathbf{u_2} \\
&\ldots\ldots\ldots \\
&\mathbf{u}_r = \mathbf{w}_r - \sum_{i=1}^{r-1} \mathrm{p}_{\mathbf{u_i}}\mathbf{w_r} = \mathbf{w}_r - \sum_{i=1}^{r-1} \frac{\langle \mathbf{w_r}, \mathbf{u_i} \rangle}{\langle \mathbf{u_i}, \mathbf{u_i} \rangle}\mathbf{u_i}.
\end{aligned}$$

By induction on i, it is possible to prove that

- $\mathbf{u}_i \in \mathbf{span}(\mathbf{w}_1, \ldots, \mathbf{w}_i)$
- $\mathbf{u}_i \neq \mathbf{0}$
- $\mathbf{u}_i$ is orthogonal to $\mathbf{u}_1, \ldots \mathbf{u}_{i-1}$.

So $\mathbf{span}(\mathbf{u}_1, \ldots, \mathbf{u}_r) = \mathbf{span}(\mathbf{w}_1, \ldots, \mathbf{w}_r)$ and $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are non zero and orthogonal, so they are linearly independent. It follows that $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ is an orthogonal basis of W. ■

Example 6.2. 1. In the space $\mathbb{R}^4$ the set

$$A = \{\mathbf{a}_1 = (1, 1, -1, 0), \mathbf{a}_2 = (0, 1, 1, 1), \mathbf{a}_3 = (2, 0, 1, 0)\}$$

is linearly independent and so the subspace $\mathbf{span}(A)$ has dimension 3. We want to find an orthonormal basis for $\mathbf{span}(A)$ using Gram-Schmidt process
$\mathbf{b}_1 = \mathbf{a}_1$,
$\mathbf{b}_2 = \mathbf{a}_2 - \mathrm{p}_{\mathbf{b}_1}\mathbf{a}_2 = (0, 1, 1, 1) - 0(1, 1, -1, 0) = (0, 1, 1, 1)$
(the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are already orthogonal),
$\mathbf{b}_3 = \mathbf{a}_3 - \mathrm{p}_{\mathbf{b}_1}\mathbf{a}_3 - \mathrm{p}_{\mathbf{b}_2}\mathbf{a}_3 = \left(\frac{5}{3}, -\frac{2}{3}, 1, -\frac{1}{3}\right)$.
The set

$$B = \left\{\mathbf{a}_1 = (1, 1, -1, 0), \mathbf{a}_2 = (0, 1, 1, 1), \mathbf{a}_3 = \left(\frac{5}{3}, -\frac{2}{3}, 1, -\frac{1}{3}\right)\right\}$$

is an orthogonal basis of $\mathbf{span}(A)$; an orthonormal basis of $\mathbf{span}$(A) is

$$C = \left\{\mathbf{c}_1 = \frac{1}{\sqrt{3}}(1, 1, -1, 0), \mathbf{c}_2 = \frac{1}{\sqrt{3}}(0, 1, 1, 1), \mathbf{c}_3 = \sqrt{\frac{3}{13}}\left(\frac{5}{3}, -\frac{2}{3}, 1, -\frac{1}{3}\right)\right\}.$$

Working with orthogonal basis is very convenient as the following result shows.

Proposition 6.3. *Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r\}$ be an orthogonal basis of the subspace $W \subset \mathbb{R}^n$ and let $\mathbf{v} \in W$. We have*

$$\mathbf{v} = \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \cdots + \lambda_r\mathbf{a}_r$$

where $\lambda_i = \dfrac{\langle \mathbf{v}, \mathbf{a}_i\rangle}{\langle \mathbf{a}_i, \mathbf{a}_i\rangle}$ for $i = 1, \ldots, r$, that is λ_i is the Fourier coefficient of $\mathbf{v}$ with respect to $\mathbf{a}_i$.

Proof
We have

$$\langle \mathbf{v}, \mathbf{a}_i\rangle = \langle \lambda_1\mathbf{a}_1 + \cdots + \lambda_i\mathbf{a}_i + \cdots + \lambda_r\mathbf{a}_r, \mathbf{a}_i\rangle = \lambda_1\langle \mathbf{a}_1, \mathbf{a}_i\rangle + \cdots + \lambda_i\langle \mathbf{a}_i, \mathbf{a}_i\rangle + \cdots + \lambda_r\langle \mathbf{a}_r, \mathbf{a}_i\rangle = \lambda_i\langle \mathbf{a}_i, \mathbf{a}_i\rangle$$

since $\langle \mathbf{a}_j, \mathbf{a}_i\rangle = 0$ if $j \neq i$. So, we get

$$\lambda_i = \frac{\langle \mathbf{v}, \mathbf{a}_i\rangle}{\langle \mathbf{a}_i, \mathbf{a}_i\rangle}, \qquad i = 1, \ldots, r. \qquad ■$$

If we take $W = \mathbb{R}^n$ and so $r = n$, for each $\mathbf{v} \in \mathbb{R}^n$, the following equality holds

$$\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{a}_1 \rangle}{\langle \mathbf{a}_1, \mathbf{a}_1 \rangle} \mathbf{a}_1 + \frac{\langle \mathbf{v}, \mathbf{a}_2 \rangle}{\langle \mathbf{a}_2, \mathbf{a}_2 \rangle} \mathbf{a}_2 + \cdots + \frac{\langle \mathbf{v}, \mathbf{a}_n \rangle}{\langle \mathbf{a}_n, \mathbf{a}_n \rangle} \mathbf{a}_n.$$

If the basis $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is orthonormal then $\langle \mathbf{a}_i, \mathbf{a}_i \rangle = 1$ and the above equality becomes

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{a}_1 \rangle \mathbf{a}_1 + \langle \mathbf{v}, \mathbf{a}_2 \rangle \mathbf{a}_2 + \cdots + \langle \mathbf{v}, \mathbf{a}_n \rangle \mathbf{a}_n.$$

In both cases we decompose the vector $\mathbf{v}$ as the sum of its orthogonal projections onto the vectors of an orthogonal (orthonormal) basis.

$$\boxed{\mathbf{v} = \mathrm{p}_{\mathbf{a}_1} \mathbf{v} + \mathrm{p}_{\mathbf{a}_2} \mathbf{v} + \cdots + \mathrm{p}_{\mathbf{a}_n} \mathbf{v}}.$$

This decomposition of $\mathbf{v}$ is called *Fourier decomposition* of $\mathbf{v}$ with respect to A.

6.4 Orthogonal projection onto a (general) subspace

Let $W = \mathbf{span}(B) \subset \mathbb{R}^n$ where $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r\}$ is an orthogonal set and so an orthogonal basis of W. As in the 1-dimensional case, given $\mathbf{v} \in \mathbb{R}^n$, we want to find the vector of W that minimizes the distance from v: this will be the best approximation of the vector $\mathbf{v}$ with a vector of W.
As before, this is the same as minimizing a function on n variables, that are the coefficients of the general vector of W expressed as linear combination of $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r$. Using the properties of the scalar product it is possible to prove that the required vector is

$$\boxed{\mathbf{v}_W = \mathrm{p}_{\mathbf{b}_1} \mathbf{v} + \mathrm{p}_{\mathbf{b}_2} \mathbf{v} + \cdots + \mathrm{p}_{\mathbf{b}_r} \mathbf{v}}.$$

This vector is called the *orthogonal projection* or simply the *projection* of $\mathbf{v}$ over W. Moreover $\mathbf{v} - \mathbf{v}_W \in W^\perp$ and is exactly the projection of $\mathbf{v}$ onto $W^\perp$. By Theorem 5.1 each subspace $W \subset \mathbb{R}^n$ determines a decomposition of $\mathbb{R}^n$ as the direct sum $\mathbb{R}^n = W \oplus W^\perp$. As a consequence each vector $\mathbf{v} \in \mathbb{R}^n$ can be written as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, with $\mathbf{v}_1 \in W$ and $\mathbf{v}_2 \in W^\perp$ in a **unique** way[3]: the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are precisely the projection of $\mathbf{v}$ onto W and $W^\perp$. We want to stress the fact that such a decomposition of $\mathbf{v}$ is unique and depends only on W: it does not depend on the orthogonal basis of W we used to find it.

Example 6.3. 1. In $\mathbb{R}^3$ let W be the subspace generated by

$$\{\mathbf{b}_1 = (1, 0, 1), \mathbf{b}_2 = (0, 1, 0)\}.$$

We want to project $\mathbf{v} = (\ 2,3,\ -1)$ onto W, that is we want to decompose $\mathbf{v} = (\ 2,3,\ -1)$ as the sum of a vector of W (the projection onto it) and

[3]Suppose that $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v} = \mathbf{v}'_1 + \mathbf{v}'_2$, with $\mathbf{v}_1, \mathbf{v}'_1 \in W$ and $\mathbf{v}_2, \mathbf{v}'_2 \in W^\perp$. Then $\mathbf{v}'_1 + \mathbf{v}'_2 = \mathbf{v}_1 + \mathbf{v}_2$ and so $\mathbf{v}_1 - \mathbf{v}'_1 = \mathbf{v}_2 - \mathbf{v}'_2 \in W \cap W^\perp$ and we get $\mathbf{v}_1 - \mathbf{v}'_1 = \mathbf{v}_2 - \mathbf{v}'_2 = \mathbf{0}$, since the sum is direct.

another vector of $W^\perp$.
The subspace W is generated by two orthogonal vectors so

$$\mathbf{v}_W = \frac{\langle \mathbf{v}, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle} \mathbf{b}_1 + \frac{\langle \mathbf{v}, \mathbf{b}_2 \rangle}{\langle \mathbf{b}_2, \mathbf{b}_2 \rangle} \mathbf{b}_2 = \frac{1}{2}(1,0,1) + 3(0,1,0) = \left(\frac{1}{2}, 3, \frac{1}{2}\right).$$

So the projection of $\mathbf{v}$ onto $W^\perp$ is

$$\mathbf{v}_{\mathbf{W}^\perp} = \mathbf{v} - \mathbf{v}_{\mathbf{W}} = (2,3,-1) - \left(\frac{1}{2}, 3, \frac{1}{2}\right) = \left(\frac{3}{2}, 0, -\frac{3}{2}\right).$$

2. In the space $\mathbb{R}^4$ we want to project the vector $\mathbf{v} = (0,0,1,1)$ onto the subspace W spanned by

$$A = \{\mathbf{a}_1 = (1,0,1,1), \mathbf{a}_2 = (0,1,2,-2), \mathbf{a}_3 = (0,0,1,-1)\}.$$

The set A is linearly independent, but it is not orthogonal. To find an orthogonal basis for W we apply the Gram-Schmidt process, obtaining the orthogonal basis

$$B = \left\{\mathbf{b}_1 = (1,0,1,1), \mathbf{b}_2 = (0,1,2,-2), \mathbf{b}_3 = \left(0, -\frac{4}{9}, \frac{1}{9}, -\frac{1}{9}\right)\right\}.$$

The orthogonal projection of $\mathbf{v} = (0,0,1,1)$ onto W is

$$\mathbf{v}_W = \mathrm{p}_{\mathbf{b}_1}\mathbf{v} + \mathrm{p}_{\mathbf{b}_2}\mathbf{v} + \mathrm{p}_{\mathbf{b}_3}\mathbf{v} = \left(\frac{2}{3}, 0, \frac{2}{3}, \frac{2}{3}\right),$$

while the projection of $\mathbf{v}$ onto $W^\perp$ is

$$\mathbf{v}_{W^\perp} = \mathbf{v} - \mathbf{v}_W = (0,0,1,1) - \left(\frac{2}{3}, 0, \frac{2}{3}, \frac{2}{3}\right) = \left(-\frac{2}{3}, 0, \frac{1}{3}, \frac{1}{3}\right).$$

3. Compute the projection of $\mathbf{v} = (1,1,0,1)$ onto the row space $\mathcal{R}(A)$ of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1 \\ -3 & 2 & 3 & 1 \end{bmatrix},$$

and onto the nullspace $\mathcal{N}$(A).
The matrix A has rank 2, and the first two rows are an orthogonal basis of $\mathcal{R}(A)$. Then

$$\mathrm{p}_{\mathcal{R}(A)} = \frac{1}{2}(1,0,-1,0) + \frac{3}{5}(0,2,0,1) = \left(\frac{1}{2}, \frac{6}{5}, -\frac{1}{2}, \frac{3}{5}\right).$$

The nullspace of A is the orthogonal complement of $\mathcal{R}(A)$, then

$$\mathrm{p}_{\mathcal{N}(A)} = (1,1,0,1) - \left(\frac{1}{2}, \frac{6}{5}, -\frac{1}{2}, \frac{3}{5}\right) = \left(\frac{1}{2}, -\frac{1}{5}, \frac{1}{2}, \frac{1}{5}\right).$$

It is worth saying that the projection of a vector onto a subspace containing it, coincides with the vector itself. Analogously, if a vector is orthogonal to a subspace its projection onto the subspace is the zero vector. That is

$$\mathbf{v}_W = \mathbf{v} \iff \mathbf{v} \in W,$$

$$\mathbf{v}_W = \mathbf{0} \iff \mathbf{v} \in W^{\perp}.$$

The definition of projection of a vector onto a subspace allows us to extend the notion of distance to the case of a vector and a subspace: we say that the *distance* from the vector $\mathbf{v}$ to the subspace W is the distance from $\mathbf{v}$ to its projection onto W

$$d(\mathbf{v}, W) = d(\mathbf{v}, \mathbf{v}_W) = \|\mathbf{v} - \mathbf{v}_W\| = \|\mathbf{v}_{W^{\perp}}\|.$$

Particularly

$$\mathrm{d}(\mathbf{v}, W) = 0 \iff \mathbf{v} \in \mathbf{W},$$
$$\mathrm{d}(\mathbf{v}, W) = ||\mathbf{v}|| \iff \mathbf{v} \in W^{\perp}.$$

Clearly it holds

$$d(\mathbf{v}, W) = \min\{d(\mathbf{v}, \mathbf{w}) \mid \mathbf{w} \in W\} = \min\{\|\mathbf{v} - \mathbf{w}\| \mid \mathbf{w} \in W\}.$$

Using this definition, in Examples 6.3, we have

1. The distance from $\mathbf{v} = (2, 3, -1)$ to $W = \mathbf{span}(\{\mathbf{b}_1 = (1, 0, 1), \mathbf{b}_2 = (0, 1, 0)\})$ is $\|\mathbf{v}_{W^{\perp}}\| = \left\|\left(\frac{3}{2}, 0, -\frac{3}{2}\right)\right\| = \frac{3}{\sqrt{2}}$.
2. The distance from $\mathbf{v} = (0, 0, 1, 1)$ to $W = \mathbf{span}\{\mathbf{a}_1 = (1, 0, 1, 1), \mathbf{a}_2 = (0, 1, 2, -2), \mathbf{a}_3 = (0, 0, 1, -1)\}$ is $\left\|\left(-\frac{2}{3}, 0, \frac{1}{3}, \frac{1}{3}\right)\right\| = \sqrt{\frac{2}{3}}$.

Example 6.4. In $\mathbb{R}^3$ we want to find the distance from $\mathbf{v} = (1, 0, -2)$ to the subspace $W = \mathbf{span}((1, 1, 0), (2, 1, -1))$.
The projection of $\mathbf{v} = (1, 0, -2)$ onto W is $\mathbf{v}_W = \left(\frac{4}{3}, -\frac{1}{3}, -\frac{5}{3}\right)$. So $\mathbf{v}_{W^{\perp}} = \mathbf{v} - \mathbf{v}_W = \left(-\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$. and $d(\mathbf{v}, W) = ||\mathbf{v}_{W^{\perp}}|| = \frac{\sqrt{3}}{3}$.

6.5 The least square solution

As discussed in Chapter 4, a real linear system $A\mathbf{x} = \mathbf{b}$ could have

1. no solutions
2. one solution
3. infinite many solutions.

By Rouché-Capelli's Theorem we are in case 1 if and only if $\text{rank}(A) < \text{rank}(A|b)$. If the system is solvable, having one solution means no free variable in the echelon form: so we are in case 2 if and only if $\text{rank}(A) = n$ and in case 3 if and only if $\text{rank}(A) < n$. In problems concerning real applications, we would like to have just one (eventually approximated) solution, even if our system has no solution or infinite many. This solution will correspond to norm minimization and will be called *the least square solution* of the system.

6.5.1 The case of a system without solutions

Given a system $A\mathbf{x} = \mathbf{b}$, with $A \in \mathbb{R}^{m\times n}$, we set $e(\mathbf{x}) = \mathbf{b} - A\mathbf{x}$ and we call it the *error vector* of $\mathbf{x} \in \mathbb{R}^n$. We cannot always get the error $e(\mathbf{x}) = \mathbf{b} - A\mathbf{x}$ down to zero: $e(\mathbf{x})$ is the zero vector if and only if $\mathbf{x}$ is an (exact) solution to $A\mathbf{x} = \mathbf{b}$ and so if and only if the system is solvable. A vector $\mathbf{x}$ such that norm of $e(\mathbf{x})$ is a small as possible is *a least square solution* of the system.
To find a least square solution, by calculus, we have to minimize the *error function* $\|e(\mathbf{x})\| = \|\mathbf{b} - A\mathbf{x}\|$ or, equivalently, the function

$$f(x_1, \dots, x_n) = \|e(\mathbf{x})\|^2 = \|\mathbf{b} - A\mathbf{x}\|^2 = \sum_{i=1}^{m} \left(b_i - \sum_{j=1}^{n} a_{ij}x_j \right)^2 ,$$

with $\mathbf{b} = (b_1, \dots, b_m)$ and $A = (a_{ij})$.

From a geometric point of view, we are minimizing the distance from $\mathbf{b}$ to the vectors of the form $A\mathbf{x}$. Since $\{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$ is exactly the column space of A, we are looking for a vector $\mathbf{x}$ realizing the distance from $\mathbf{b}$ to the subspace $\mathcal{C}(A)$. Since

$$d(\mathbf{b}, \mathcal{C}(A)) = d\left(\mathbf{b}, \text{p}_{\mathcal{C}(A)}\mathbf{b}\right),$$

we are looking for a vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \text{p}_{\mathcal{C}(A)}\mathbf{b}$. So, a least square solution of $A\mathbf{x} = \mathbf{b}$ is a solution of the system $A\mathbf{x} = \text{p}_{\mathcal{C}(A)}\mathbf{b}$. This system is always solvable, since, by definition, $\text{p}_{\mathcal{C}(A)}\mathbf{b} \in \mathcal{C}(A)$. Unfortunately, finding $\text{p}_{\mathcal{C}(A)}b$ could require a lot of computations, because to apply the formula of pag. 71 we must have an orthogonal basis for $\mathcal{C}(A)$. Next result will give us another way to find a least square solution.

Proposition 6.4. *The system $A\mathbf{x} = \text{p}_{\mathcal{C}(A)}b$ is equivalent to the system $A^TA\mathbf{x} = A^T\mathbf{b}$.*

Proof We can write $\mathbf{b}$ as $\mathbf{b} = A\mathbf{v} + (\mathbf{b} - A\mathbf{v})$, for any $\mathbf{v} \in \mathbb{R}^n$. By Proposition 5.5 we have $\mathbb{R}^m = \mathcal{C}(A) \oplus \mathcal{N}(A^T)$ and so we can decompose $\mathbf{b}$ as $\mathbf{b} = \mathbf{b}' + \mathbf{b}''$ with $\mathbf{b}' = \text{p}_{\mathcal{C}(A)}b \in \mathcal{C}(A)$ and $\mathbf{b}'' = \text{p}_{\mathcal{N}(A^T)}b \in \mathcal{N}(A^T)$. Since such a decomposition is unique, $\mathbf{v}$ is a solution of $A\mathbf{x} = \text{p}_{\mathcal{C}(A)}b$, if and only if

$$b' = A\mathbf{v} \iff \mathbf{b} - A\mathbf{v} = b'' \iff \mathbf{0} = A^T(\mathbf{b} - A\mathbf{v}) = A^T\mathbf{b} - A^TA\mathbf{v}.$$

Clearly this holds if and only if $\mathbf{v}$ is a solution of the system $A^TA\mathbf{x} = A^T\mathbf{b}$. ■

So finding an (approximated) least square solution of a system $A\mathbf{x} = \mathbf{b}$ is equivalent to finding an exact solution of the system

$$A^TA\mathbf{x} = A^T\mathbf{b} \tag{1}$$

called *the associated normal system*. Notice that if $A\mathbf{x} = \mathbf{b}$ is solvable then $\mathbf{b} \in \mathcal{C}(A)$ and so $\mathrm{p}_{\mathcal{C}(A)}\mathbf{b} = \mathbf{b}$: in this case the normal system is a square system equivalent to the starting one. The coefficient matrix A^TA of the normal system is called the *Gram matrix by columns* of A: it is straightforward to check that the ij entry of A^TA is equal to the scalar product between the i-th column and the j-th column of A. As a consequence A^TA is a symmetric matrix of order n. There is a deep relation between the matrices A and A^TA as the following result shows.

Proposition 6.5. *Let $A \in \mathbb{R}^{m\times n}$, then*

1. $\mathcal{N}(A) = \mathcal{N}(A^TA)$,
2. $\mathrm{rank}(A) = \mathrm{rank}(A^TA)$,
3. $\mathrm{rank}(A) = n \iff A^TA$ is non singular,
4. $\mathcal{R}(A) = \mathcal{R}(A^TA)$.

Proof

1. We have $\mathbf{x} \in \mathcal{N}(A) \iff A\mathbf{x} = \mathbf{0} \implies A^TA\mathbf{x} = A^T\mathbf{0} = \mathbf{0} \implies \mathbf{x} \in \mathcal{N}(A^TA)$.
   On the contrary $\mathbf{x} \in \mathcal{N}(A^TA) \iff (A^TA)\mathbf{x} = \mathbf{0}$ and so $\mathbf{x}^TA^TA\mathbf{x} = \mathbf{x}^T\mathbf{0} = 0$. Since $\mathbf{x}^TA^TA\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x}) = \langle A\mathbf{x}, A\mathbf{x}\rangle = \|A\mathbf{x}\|^2 = 0$ we have $A\mathbf{x} = \mathbf{0}$ and so $\mathbf{x} \in \mathcal{N}(A)$.
2. Denote with r the rank of A and with k the rank of A^TA. We have
   $$n - r = \dim(\mathcal{N}(A)) = \dim(\mathcal{N}(A^TA)) = n - k \implies r = k.$$
3. It follows directly from the previous point.
4. By Proposition 5.5, we have $\mathcal{N}(A^TA)^\perp = \mathcal{R}(A^TA)$ and $\mathcal{N}(A)^\perp = \mathcal{R}(A)$. Then $\mathcal{R}(A^TA)$ and $\mathcal{R}(A)$ coincide, since they both are the orthogonal complement of the same subspace. ■

There is also a definition of *Gram matrix by rows* of A: it is the symmetric order n matrix given by $AA^T \in \mathbb{R}^{n\times n}$. Its entries are the scalar product between the rows of A.

If we look at all the possible solutions of the normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ only two cases arise

1. the system has only one solution: this happens if and only if the matrix A^TA is non singular and so if and only if the rank of A is n, (i. e. A has linearly independent columns).

2. the system has infinite many solution: this happens if and only if the matrix A^TA is singular and so if and only if the rank of A is less then n, (i. e. A has linearly dependent columns).

In the first case we have only one least square solution and so we call it **the** least square solution. Moreover we have an explicit formula for it

$$\boxed{\mathbf{x}^* = (A^TA)^{-1}A^T\mathbf{b}.}$$

Example 6.5. The linear system

$$A\mathbf{x} = \begin{bmatrix} 3 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{b}$$

has no solution since the matrix A has rank 2, while the rank of $[A|\mathbf{b}]$ is 3. The associated normal system is

$$A^TA\mathbf{x} = \begin{bmatrix} 10 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = A^T\mathbf{b},$$

and has only one solution that is

$$\begin{bmatrix} x \\ y \end{bmatrix} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{3}{26} & -\frac{1}{13} \\ -\frac{1}{13} & \frac{5}{13} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{13} \\ \frac{8}{13} \end{bmatrix}.$$

The vector $\mathbf{x}^* = \left(\frac{1}{13}, \frac{8}{13}\right)$ is the least square solution of $A\mathbf{x} = \mathbf{b}$. This means that $A\mathbf{x}^* = \left(\frac{11}{13}, \frac{8}{13}, \frac{7}{13}\right)$ is the projection of $\mathbf{b}$ onto $\mathcal{C}(A)$. Moreover

$$\|\mathbf{b} - A\mathbf{x}^*\| = \left\| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{11}{13} \\ \frac{8}{13} \\ \frac{7}{13} \end{bmatrix} \right\| \approx 0,78$$

is the minimum value of the error function $\|e(\mathbf{x})\| = \|\mathbf{b} - A\mathbf{x}\|$.

If the normal system has infinite many solutions, we want to find the "best one": in the following section we will explain, more generally, what is the meaning of finding the "best solution" of a liner system having infinite many solutions.

6.5.2 The case of a system with infinite many solutions

Let $C\mathbf{x} = \mathbf{d}$ be a linear system, with m equations and n unknowns, such that $\text{rank}(C) = \text{rank}(C|\mathbf{d}) = r$. The set of the solutions $L = \{\mathbf{x} \in \mathbb{R}^n \mid C\mathbf{x} = \mathbf{d}\}$ is an affine subspace of $\mathbb{R}^n$; if $r = n$ it contains only one vector, while if $r < n$ it contains infinite many vectors. As we already stressed, in applications, as the search for a linear statistical model (for example a linear regression) we are interested in having *only one* solution. So, in the case $r < n$, we have to decide how to choose a particular solution between the infinite many. A well-known criteria is to choose the one having minimum norm: the efficiency of such a criteria relies on the following statement.

Theorem 6.2. *Let $C\mathbf{x} = \mathbf{d}$ be a solvable linear system then*

1. *for any two solutions $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$ we have $\mathrm{p}_{\mathcal{R}(C)}\mathbf{x} = \mathrm{p}_{\mathcal{R}(C)}\mathbf{x}'$.*
2. *If we set $\mathbf{x}_\mathcal{R} = \mathrm{p}_{\mathcal{R}(C)}\mathbf{x}$, where $\mathbf{x}$ is any solution, then $\mathbf{x}_\mathcal{R}$ is itself a solution of the system. Moreover*
3. *for any solution $\mathbf{x}$ of the system we have $\|\mathbf{x}_\mathcal{R}\| \leq \|\mathbf{x}\|$.*

Proof By Proposition 5.5 we have that $\mathcal{N}(C)^\perp = \mathcal{R}(C)$ and so $\mathbb{R}^n = \mathcal{R}(C) \oplus \mathcal{N}(C)$. Given two solutions $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$ we can decompose them as $\mathbf{x} = \mathbf{x}_\mathcal{R} + \mathbf{x}_\mathcal{N}$ and $\mathbf{x}' = \mathbf{x}'_\mathcal{R} + \mathbf{x}'_\mathcal{N}$, where $\mathbf{x}_\mathcal{R}, \mathbf{x}'_\mathcal{R} \in \mathcal{R}(C)$ are the projections of $\mathbf{x}, \mathbf{x}'$ onto $\mathcal{R}(C)$ and $\mathbf{x}_\mathcal{N}, \mathbf{x}'_\mathcal{N} \in \mathcal{N}(C)$ are the projections of $\mathbf{x}, \mathbf{x}'$ onto $\mathcal{N}(C)$. We have

$$\begin{aligned} \mathbf{d} &= C\mathbf{x} = C(\mathbf{x}_\mathcal{R} + \mathbf{x}_\mathcal{N}) = C\mathbf{x}_\mathcal{R} + C\mathbf{x}_\mathcal{N} = C\mathbf{x}_\mathcal{R} + \mathbf{0} = C\mathbf{x}_\mathcal{R} \\ \mathbf{d} &= C\mathbf{x}' = C(\mathbf{x}'_\mathcal{R} + \mathbf{x}'_\mathcal{N}) = C\mathbf{x}'_\mathcal{R} + C\mathbf{x}'_\mathcal{N} = C\mathbf{x}'_\mathcal{R} + \mathbf{0} = C\mathbf{x}'_\mathcal{R}. \end{aligned}$$

From $\mathbf{d} = C\mathbf{x}_\mathcal{R}$ we get that $\mathbf{x}_\mathcal{R}$ is a solution of the system. Moreover, from $C\mathbf{x}_\mathcal{R} = C\mathbf{x}'_\mathcal{R}$ we obtain $C(\mathbf{x}_\mathcal{R} - \mathbf{x}'_\mathcal{R}) = \mathbf{0}$ and so $\mathbf{x}_\mathcal{R} - \mathbf{x}'_\mathcal{R} \in \mathcal{N}(C)$. Clearly $\mathbf{x}_\mathcal{R} - \mathbf{x}'_\mathcal{R} \in \mathcal{R}(C)$, therefore $\mathbf{x}_\mathcal{R} - \mathbf{x}'_\mathcal{R} \in \mathcal{R}(C) \cap \mathcal{N}(C)$. Since the only element of the intersection is the zero vector we obtain $\mathbf{x}_\mathcal{R} - \mathbf{x}'_\mathcal{R} = \mathbf{0}$ and so $\mathbf{x}_\mathcal{R} = \mathbf{x}'_\mathcal{R}$. The last statement follows from the Pythagora's Theorem applied to the decomposition $\mathbf{x} = \mathbf{x}_\mathcal{R} + \mathbf{x}_\mathcal{N}$

$$\|\mathbf{x}\|^2 = \|\mathbf{x}_\mathcal{R} + \mathbf{x}_\mathcal{N}\|^2 = \|\mathbf{x}_\mathcal{R}\|^2 + \|\mathbf{x}_\mathcal{N}\|^2 \geq \|\mathbf{x}_\mathcal{R}\|^2$$

that implies $\|\mathbf{x}\| \geq \|\mathbf{x}_\mathcal{R}\|$. ■

Example 6.6.

Consider the system

$$\begin{cases} x - y + z - t = 1 \\ y + t = 0 \\ 2x - 5y + 2z - 5t = 2 \end{cases},$$

it is solvable since the coefficient and the augmented matrix have both rank 2 as we can see finding a row echelon form

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & -5 & 2 & -5 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The set of solutions is

$$L = \{(1-\alpha, -\beta, \alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\} \subset \mathbb{R}^4.$$

The minimum norm solution is the projection of any vector of L onto the row space of

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & -5 & 2 & -5 \end{bmatrix}.$$

From the basis of $\mathcal{R}(A)$

$$\{(1,-1,1,-1),(0,1,0,1)\},$$

via the Gram-Schmidt process, we obtain the orthogonal basis

$$\{(1,-1,1,-1),(1,1,1,1\}.$$

Given a vector of L, for example $\mathbf{v} = (0,0,1,0)$ (obtained for $\alpha = 1$ and $\beta = 0$), the minimum norm solution is the projection of $\mathbf{v}$ onto $\mathcal{R}(A)$ and so it is $\mathbf{v}_{\mathcal{R}} = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$.

Definition 6.1. *The least square solution* of a linear system $A\mathbf{x} = \mathbf{b}$ is the least norm solution of the associated normal system $A^T A\mathbf{x} = A\mathbf{b}$. Since $\mathcal{R}(A^T A) = \mathcal{R}(A)$, it can be obtained by projecting any solution of the normal system onto the row space of A.

Notice that in the case of a solvable system, it is not necessary to compute the normal system: it is enough to find any solution of the system and to project it onto the row space.

Each system admits a unique least square solution which can be either an exact solution (in the case of a solvable system) or an approximated one (in the case of an unsolvable system). Moreover, the least square solution is obtained by minimizing twice the Euclidean norm (which is a sum of squares): first we find all the vectors which minimize $\|\mathbf{b} - A\mathbf{x}\|$, then, among them, we find the one of minimum norm.

Example 6.7. The linear system

$$A\mathbf{x} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

is unsolvable: the matrix A has rank 2, while $\operatorname{rank}[A\mathbf{b}] = 3$. The normal system is

$$A^T A\mathbf{x} = \begin{bmatrix} 10 & 0 & 10 \\ 0 & 5 & -5 \\ 10 & -5 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^T\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

The coefficient matrix A^TA is singular, since it has the same rank of A; so the normal system has infinite many solutions. To find them we use Gauss algorithm and obtain the equivalent echelon system

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 = \dfrac{1}{5} \end{cases},$$

whose solutions are

$$L = \left\{ \left(-\alpha, \frac{1}{5} + \alpha, \alpha \right) \mid \alpha \in \mathbb{R} \right\}.$$

To find the minimum norm solution we can project any vector of L onto the row space, $\mathcal{R}(A)$, of A.
An orthonormal basis of the row space is

$$\left\{ \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left(-\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right) \right\},$$

and so the projection onto $\mathcal{R}(A)$ of the solution, for example $\mathbf{v} = \left(0, \frac{1}{5}, 0\right)$ is $\mathbf{v}_{\mathcal{R}} = \left(\frac{1}{15}, \frac{2}{15}, -\frac{1}{15} \right)$: so this vector is the least square solution of the initial system.

Remark The least square solution of an impossible system is, generally, not invariant under row operation on the augmented matrix. For example, consider the 1-variable systems

$$S: \begin{cases} x = 0 \\ x = 1 \end{cases} \qquad S': \begin{cases} 2x = 0 \\ x = 1 \end{cases}.$$

The two augmented matrices are clearly row equivalent, but it is easy to check that the least square solution of S is $x^* = \frac{1}{2}$ while the one of S' is $x^* = \frac{2}{5}$. This is due to the fact that row operations change the columns space and so the projection of $\mathbf{b}$ onto it. For the same reason, also the removal of an equation which is linear combination of the others usually changes the least square solution. For example, the least square solution of the system

$$S'': \begin{cases} x = 0 \\ x = 1 \\ x = 1 \end{cases}$$

is

$$x^* = \left((1,1,1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)^{-1} (1,1,1) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{2}{3},$$

and so it is different from the one of S.

6.6 Projection matrices

If the coefficient matrix of a linear system has linearly independent columns we have an explicit formula to find the least square solution. Using it, we have a different way to find the projection of a vector onto a subspace.

Proposition 6.6. *Let $W \subset \mathbb{R}^n$ be a subspace. Choose a basis $\mathcal{B}$ of W and denote with A the matrix having as columns the vectors of $\mathcal{B}$. For any $\mathbf{v} \in \mathbb{R}^n$ we have*

$$\mathrm{p}_W(\mathbf{v}) = A(A^TA)^{-1}A^T\mathbf{v}.$$

Proof The column of A are linearly independent, so the normal system associated to $A\mathbf{x} = \mathbf{v}$ has only one solution given by $\mathbf{x}^* = (A^TA)^{-1}A^T\mathbf{v}$. Since the normal system is equivalent to $A\mathbf{x} = \mathrm{p}_{\mathcal{C}(A)}\mathbf{v}$ we have $A(A^TA)^{-1}A^T\mathbf{v} = \mathrm{p}_{\mathcal{C}(A)}\mathbf{v} = \mathrm{p}_W\mathbf{v}$. ■

The matrix $P_W = A(A^TA)^{-1}A^T \in \mathbb{R}^{n\times n}$ does not depend on the basis $\mathcal{B}$ that we use to compute it and is called *the projection matrix* onto W. If we use an **orthonormal basis** $\mathcal{B}$ of W, the columns of the matrix A are orthonormal and so $A^TA = I_n$ (recall that the entry ij of the Gram matrix by columns is the the inner product of the i-th and j-th column of A). In this case, the formula of the projection matrix simplifies to $P_W = AA^T$, so P_W is the Gram matrix by rows of A.

It is easy to check that a projection matrix P onto a subspace $W \subset \mathbb{R}^n$ has the following properties

1. $P^2 = P \cdot P = P$ (we say that P is *idempotent*)
2. P is symmetric
3. $\mathrm{rank}(P) = \dim W$
4. $I_n - P$ is the projection matrix onto $W^\perp$.

The properties 1 and 2 characterize the projection matrices: if $P \in \mathbb{R}^{n\times n}$ is an idempotent symmetric matrix then it is the projection matrix onto its column space $\mathcal{C}(P)$. Therefore, there is a bijection between the set of subspaces of $\mathbb{R}^n$ and the set of idempotent symmetric matrices of $\mathbb{R}^{n\times n}$.

Example 6.8. 1. We want to construct the projection matrix onto the subspace $W = \mathbf{span}(\mathbf{a}_1 = (1,0,-1), \mathbf{a}_2 = (0,1,2), \mathbf{a}_3 = (1,1,1)) \subset \mathbb{R}^3$. A basis of W is $\{\mathbf{a}_1, \mathbf{a}_2\}$, since $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$. We construct the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$$

and the projection matrix is

$$P_W = A(A^TA)^{-1}A^T = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix}.$$

So, fox example, the projection of $\mathbf{v} = (1, 0, 0)$ onto W is

$$P_W(\mathbf{v}) = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{1}{3} \\ -\frac{1}{6} \end{bmatrix}.$$

2. We want to find the projection matrix over the subspace $W = \mathbf{span}(\{\mathbf{b}_1 = (1, 0, 2, -1), \mathbf{b}_2 = (0, 1, 1, 0)\}) \subset \mathbb{R}^4$.
   We construct the matrix B having $\mathbf{b}_1$ and $\mathbf{b}_2$ as columns

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \\ -1 & 0 \end{bmatrix},$$

   so the projection matrix onto W is

$$P_W = B(B^TB)^{-1}B^T = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

3. In $\mathbb{R}^n$, we want to compute the projection matrix onto the subspace spanned by the vector

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

   Normalizing the vector we get

$$\begin{bmatrix} \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{bmatrix},$$

so the projection matrix is

$$P = \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{bmatrix} = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}.$$

The matrix $C_n = I_n - P$, still symmetric and idempotent, is the projection matrix onto $\mathbf{1}^\perp$: in statistics, it is usually called the *centering matrix*, since $C_n\mathbf{v}$ is the vector obtained by subtracting from each component of $\mathbf{v}$ the arithmetic mean of all its components. For example, if $n = 3$, the centering matrix is

$$C_3 = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

For a vector $\mathbf{v} = (x_1, x_2, x_3)$, let $\mu = \dfrac{x_1 + x_2 + x_3}{3}$ be the average of its components; the vector $C_3\mathbf{v}$ is

$$C_3\mathbf{v} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - \mu \\ x_2 - \mu \\ x_3 - \mu \end{bmatrix}.$$

4. If W is the one dimensional subspace spanned by a vector $\mathbf{b} = (b_1, \ldots, b_n)$, the projection matrix is the rank 1 matrix

$$P_{\mathbf{b}} = \frac{1}{\mathbf{b}^T\mathbf{b}} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \ldots & b_n \end{bmatrix} = \frac{1}{\|\mathbf{b}\|^2} \begin{bmatrix} b_1^2 & b_1b_2 & \ldots & b_1b_n \\ b_2b_1 & b_2^2 & \ldots & b_2b_n \\ \vdots & \vdots & \vdots & \vdots \\ b_nb_1 & b_nb_2 & \ldots & b_n^2 \end{bmatrix}.$$

We end this chapter with a particular case of least square solution. Let $A\mathbf{x} = \mathbf{b}$ be a linear system such that **the rows** of $A \in \mathbb{R}^{m\times n}$ are linearly independent. We have

$$m = \operatorname{rank}(A) = \dim \mathcal{C}(A) \subset \mathbb{R}^m \Longrightarrow \mathcal{C}(A) = \mathbb{R}^m,$$

so the system is solvable. Moreover, the (exact) least square solution is

$$\boxed{\mathbf{x}^* = A^T(AA^T)^{-1}\mathbf{b}.}$$

In fact

$$A\mathbf{x}^* = AA^T(AA^T)^{-1}\mathbf{b} = I_m\mathbf{b} = \mathbf{b},$$

so $\mathbf{x}^*$ is a solution, and then

$$x^* = A^T(AA^T)^{-1}\mathbf{b} = A^T(AA^T)^{-1}A\mathbf{x}^*.$$

But $A^T(AA^T)^{-1}A$ is exactly the projection matrix $P_{\mathcal{R}(A)}$ onto $\mathcal{R}(A)$, so we have $x^* = P_{\mathcal{R}(A)}x^*$ that implies $x^* \in \mathcal{R}(A)$.

Remark The notation commonly used in the statistical analysis of data for a linear system (used, for example, to find a linear regression) is $X\boldsymbol{\beta} = \mathbf{y}$, instead of $A\mathbf{x} = \mathbf{b}$. The coefficient matrix X is called the *design matrix*, β is the *parameter vector* and $\mathbf{y}$ is the *observation vector*. The error vector $\mathbf{y} - X\beta$ is also called the *residual vector*. If X is full rank by columns, the projection matrix $\widehat{X} = X(X^TX)^{-1}X^T$ onto $\mathcal{C}(X)$ is called the *hat matrix*.

Chapter 7

7 Linear transformations

7.1 General definition

Let $\mathbf{V}, \mathbf{W}$ be vector spaces over the real numbers. A function

$$f : \mathbf{V} \longrightarrow \mathbf{W}$$

is called a *linear transformation* if the following two properties hold

1. $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}), \qquad \text{for all} \quad \mathbf{u}, \mathbf{v} \in \mathbf{V}$
2. $f(\lambda \mathbf{u}) = \lambda f(\mathbf{u}), \qquad \text{for all} \quad \lambda \in \mathbb{R}, \mathbf{u} \in \mathbf{V}.$

As consequences of the above properties we have

- $f(\lambda \mathbf{u} + \mu \mathbf{v}) = \lambda f(\mathbf{u}) + \mu f(\mathbf{v}), \qquad \text{for all} \quad \lambda, \mu \in \mathbb{R}, \mathbf{u}, \mathbf{v} \in \mathbf{V}$
- $f(\mathbf{0_V}) = \mathbf{0_W}$.

An *endomorphism* of $\mathbf{V}$ is a linear transformation from $\mathbf{V}$ to itself; a bijective linear transformation is called *isomorphism*.

Example 7.1.

1. Let $a \in \mathbb{R}$, with $a \neq 0$. The function $\lambda_a : \mathbf{V} \to \mathbf{V}$ defined by $\lambda_a(\mathbf{v}) \to a\mathbf{v}$, rescaling each vector by a factor a, is an isomorphism and is called *homothetic transformation* or *homothety* of ratio a. The function $\mathbf{0} : \mathbf{V} \to \mathbf{W}$ sending all the vectors of $\mathbf{V}$ into the zero vector of $\mathbf{W}$ is linear and is called the *null transformation*. Clearly it is not an isomorphism.

2. Let $\mathbf{b} \in \mathbb{R}^n$ be a non zero vector. The translation $t_{\mathbf{b}} : \mathbf{V} \to \mathbf{V}$ given by $t_{\mathbf{b}}(\mathbf{v}) = \mathbf{v} + \mathbf{b}$ is not a linear transformation since $t_{\mathbf{b}}(\mathbf{0}) = \mathbf{b} \neq \mathbf{0}$.

3. Let $\mathbb{R}^{m \times n}$ be the vector space of the matrices $m \times n$ with real coefficient. The function
$$T : \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}^{n \times m} \qquad A \mapsto A^T$$
is an isomorphism. On the contrary, the determinant $\det : \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$ is not a linear transformation.

4. Let $\mathcal{C}^\infty(\mathbb{R})$ be the set of real smooth functions in one real variable: function addition and multiplication by real numbers gives to $\mathcal{C}^\infty(\mathbb{R})$ the structure of vector space. The first derivative
$$D : \mathcal{C}^\infty(\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}), \qquad D(f) = f'$$
is an endomorphism (which is not an isomorphism).

5. The transformation $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = x + y$ is linear, while $F : \mathbb{R}^2 \to \mathbb{R}$ given by $F(x, y) = x^2 + y$ and $G : \mathbb{R}^2 \to \mathbb{R}$ defined as $F(x, y) = y + 1$ are not. In the next section, we completely characterize the linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$.

It is straightforward that

- if g, f are linear transformations such that $f \circ g$ is defined, then $f \circ g$ is linear
- the identity function $\mathbf{1_V} : \mathbf{V} \to \mathbf{V}$, sending each vector to itself, is linear
- if f is an isomorphism (i.e. linear and bijective) then f^{-1} is an isomorphism.

Let $f : \mathbf{V} \to \mathbf{W}$ be a linear transformation we define

$\ker f = \{\mathbf{v} \in \mathbf{V} \mid f(\mathbf{v}) = \mathbf{0_W}\} = f^{-1}(\mathbf{0_W}) \subset \mathbf{V}$ and call it the *kernel* of f

$\mathrm{im} f = \{f(\mathbf{v}) \in \mathbf{W} \mid \mathbf{v} \in \mathbf{V}\} = f(\mathbf{V}) \subset \mathbf{W}$ and call it the *range* or the *image* of f.

Both the kernel and the range are subspaces of the respective vector spaces. Moreover, we have the following result.

Theorem 7.1. *Let $f : \mathbf{V} \to \mathbf{W}$ be a linear transformation between vector spaces.*

1. *f is injective if and only if $\ker f = \{\mathbf{0_V}\}$ if and only if $\dim(\ker f) = 0$ if and only if for each finite independent linear set X, the set $f(X)$ is independent*
2. *f is surjective if and only if $\mathrm{im} f = \mathbf{W}$ if and only if for each finite set of generators X of $\mathbf{V}$, the set $f(X)$ spans $\mathbf{W}$. If $\mathbf{W}$ is finitely generated, the previous statements are also equivalent to $\dim(\mathrm{im} f) = \dim \mathbf{W}$*
3. *if $\mathbf{V}$ is finitely generated then*

$$\dim(\ker f) + \dim(\mathrm{im} f) = \dim \mathbf{V} \qquad \text{(rank-nullity theorem).}$$

As a corollary two finitely generated vector spaces are isomorphic if and only if they have the same dimension.

Theorem 7.2. *Every real vector space $\mathbf{V}$ of dimension n is isomorphic to $\mathbb{R}^n$.*

Proof Let $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an ordered basis of $\mathbf{V}$ (a basis with a fixed order on the vectors). Since $\mathcal{B}$ spans $\mathbf{V}$, for each $\mathbf{v} \in \mathbf{V}$ there exist n scalars $a_1, \ldots, a_n$ such that $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. The fact that $\mathcal{B}$ is linearly independent implies that such a decomposition is unique: that is if $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$ then $a_i = b_i$ for each $i = 1, \ldots, n$. So the function $\varphi_{\mathcal{B}} : \mathbf{V} \to \mathbb{R}^n$ defined by $\varphi_{\mathcal{B}}(\mathbf{v}) = (a_1, \ldots, a_n)$ is well-defined. Moreover it is really easy to check that it is an isomorphism. ■

The previous result explains why the study of $\mathbb{R}^n$ plays a central role at least in the theory of finite dimensional vector spaces: for this reason $\mathbb{R}^n$ is also called the *standard n-dimensional vector space*. Unfortunately, many important vector spaces such as the spaces of functions are not finitely generated: nevertheless also in this case we can sometimes work in a finite dimensional setting, and so in $\mathbb{R}^n$, up to "approximations".

7.2 Linear transformations between $\mathbb{R}^n$ and $\mathbb{R}^m$

In the case of functions between standard spaces, linear transformations can be completely characterized.

Theorem 7.3. *Let $A \in \mathbb{R}^{m\times n}$. The function*

$$f_A : \mathbb{R}^n \to \mathbb{R}^m \qquad f_A(\mathbf{x}) = A\mathbf{x}$$

is a linear transformation. Conversely, let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation then there exist a unique matrix $A \in \mathbb{R}^{m\times n}$ such that $f = f_A$. As a consequence, any linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ has the form

$$f(x_1, \ldots, x_n) = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (a_{11}x_1 + \ldots a_{1n}x_n, \ldots, a_{m1}x_1 + \ldots a_{mn}x_n),$$

that is, the components of $f(x_1, \ldots, x_n)$ are homogeneous first degree polynomial in the unknowns $x_1, \ldots, x_n$.

Proof Let $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Using the properties of matrix multiplication we have

$$\begin{aligned} f_A(\mathbf{x} + \mathbf{x}') &= A(\mathbf{x} + \mathbf{x}') = A\mathbf{x} + A\mathbf{x}' = f_A(\mathbf{x}) + f_A(\mathbf{x}') \\ f_A(\lambda\mathbf{x}) &= A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda f_A(\mathbf{x}), \end{aligned}$$

that is f_A is a linear transformation. Conversely, suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and consider the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$. Let A be the matrix having has i-th column the vector $f(\mathbf{e}_i)$: that is $A = (a_{ij})$ with $f(\mathbf{e}_i) = (a_{1i}, \ldots, a_{mi})$, for $i = 1, \ldots, n$. Given any $\mathbf{v} \in \mathbb{R}^n$, we want to prove that $f(\mathbf{v}) = A\mathbf{v}$. If $\mathbf{v} = (v_1, \ldots, v_n)$ then $\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n$. We have

$$\begin{aligned} f(\mathbf{v}) &= f(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 \cdots + v_n\mathbf{e}_n) = v_1 f(\mathbf{e}_1) + v_2 f(\mathbf{e}_2) \cdots + v_n f(\mathbf{e}_n) = \\ &= v_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + v_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots + v_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \\ &= \begin{pmatrix} v_1a_{11} + v_2a_{12} + \cdots + v_na_{1n} \\ v_1a_{21} + v_2a_{22} + \cdots + v_na_{2n} \\ \vdots \\ v_1a_{m1} + v_2a_{m2} + \cdots + v_na_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}. \end{aligned}$$

Suppose that there are two matrices $A, B \in \mathbb{R}^{m\times n}$ such that $f = f_A = f_B$. We have

$$f_A(\mathbf{x}) = A\mathbf{x} = B\mathbf{x} = f_B(\mathbf{x})$$

for each $\mathbf{x} \in \mathbb{R}^n$. Particularly $A\mathbf{e}_i = B\mathbf{e}_i$, that is, the i-th column of A is equal to the i-th column of B, for $i = 1, \ldots, n$ and so $A = B$. ■

If $f = f_A$, we say that A *(canonically) represent* the linear transformation f. So, there is a (canonical) bijective correspondence between the set (a vector space) of all the linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ and (the vector space) $\mathbb{R}^{m\times n}$. The adjective canonical has to deal with the fact that the correspondence is constructed via the standard bases. In the next section we will investigate what happens if we try to construct such a correspondence bases different from the standard ones.

The possibility of describing linear transformations via matrices, as well as the next result, motivates the "apparently unnatural" definition of matrix multiplication.

Proposition 7.1. *Under the correspondence between linear transformations and matrices, the composition between linear transformation corresponds to the product between matrices, that is*

- *if* $f_A : \mathbb{R}^m \to \mathbb{R}^n$ *and* $f_B : \mathbb{R}^n \to \mathbb{R}^p$ *then* $f_{BA} = f_B \circ f_A : \mathbb{R}^m \to \mathbb{R}^p$
- f_{I_n} *is the identity function* $\mathbf{1}_n$ *of* $\mathbb{R}^n$
- $f_C : \mathbb{R}^n \to \mathbb{R}^n$ *is invertible if and only if* C *is invertible and* $f_C^{-1} = f_{C^{-1}}$.

Furthermore, $\ker(f_A) = \mathcal{N}(A)$ *and* $\operatorname{im}(f_A) = \mathcal{C}(A)$.

Note that the null-rank theorem for f_A corresponds to the decomposition of Proposition 5.5. Moreover, a linear transformation f_A is

- injective if and only if the nullspace is the trivial subspace of $\mathbb{R}^n$ if and only if $\dim(\mathcal{N}(A)) = n - \operatorname{rank}(A) = 0$, that is $n = \operatorname{rank}(A)$
- surjective if and only if the column space it the whole target space $\mathbb{R}^m$ if and only if $\dim(\mathcal{C}(A)) = \operatorname{rank}(A) = m$
- bijective if and only if $m = n = \operatorname{rank}(A)$ that is if and only if A is regular.

Example 7.2. 1. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be the linear transformation defined by

$$f(x, y, z) = (x - 3y, x + y + 4z, z, x - 3y + 5z).$$

We have $f(1, 0, 0) = (1, 1, 0, 1)$, $f(0, 1, 0) = (-3, 1, 0, -3)$ and $f(0, 0, 1) = (0, 4, 1, 5)$. So

$$f(\mathbf{x}) = f_A(\mathbf{x}) = A\mathbf{x} = \begin{pmatrix} 1 & -3 & 0 \\ 1 & 1 & 4 \\ 0 & 0 & 1 \\ 1 & -3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We have $\text{rank}(A) = 3$ and so f is injective but not surjective: $\text{im}f$ is a 3-dimensional subspace of $\mathbb{R}^4$ having the columns of A as a basis.

2. The matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ defines the linear transformation

$$f_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad f_A(\mathbf{x}) = A\mathbf{x} = (x_1 + x_3, 2x_2 + 2x_3).$$

The kernel of f_A

$$\ker f_A = \mathcal{N}(A) = \{(-\alpha, -\alpha, \alpha) \mid \alpha \in \mathbb{R}\},$$

has dimension one; so the dimension of the range of f_A, that is the rank of A, is $3 - 1 = 2$. This means that f_A is surjective.

3. The matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ is associated to the endomorphism of $\mathbb{R}^3$

$f_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad f_A(x_1, x_2, x_3) = (x_1+3x_2+3x_3, x_1+4x_2+3x_3, x_1+3x_2+4x_3).$

The matrix A is non-singular and its inverse is

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

So the endomorphism f_A is bijective (i.e. it is an isomorphism) and its inverse is $f_A^{-1} = f_{A^{-1}} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$f_A^{-1}(y_1, y_2, y_3) = A^{-1} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = (7y_1 - 3y_2 - 3y_3, -y_1 + y_2, -y_1 + y_3).$$

We end this section describing endomorphisms associated to particular class of matrices.

7.2.1 Linear isometry and orthogonal matrices

An endomorphism

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is called an *orthogonal transformation* or a *linear isometry* if

$$\langle f(\mathbf{v}_1), f(\mathbf{v}_2) \rangle = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle, \quad \text{for all} \quad \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n,$$

that is if the inner product is an *invariant* with respect to f. As a consequence, an orthogonal transformation keep lengths, distances and angles unchanged. In other words orthogonal transformations are "geometrical isometries" having at least a fixed point (that is the zero vector).

A matrix $A \in \mathbb{R}^n$ is said *orthogonal* if it has orthonormal columns i.e. $A^T A = I_n$. It is possible to prove that this happens if and only if it has orthonormal rows i. e $AA^T = I_n$. Equivalently, A is invertible and $A^{-1} = A^T$. The set of orthogonal matrices of order n is denoted with $\mathcal{O}_n$. Under the standard correspondence between endomorphisms over $\mathbb{R}^n$ and matrices of $\mathbb{R}^{n\times n}$, orthogonal transformations correspond to orthogonal matrices.

Example 7.3. The matrix $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathbb{R}^{2\times 2}$ and the corresponding linear isometry $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation around the origin by the angle θ.

It is easy to check that the composition of two orthogonal transformations, as well as the product of two orthogonal matrices, is still orthogonal.

7.2.2 Symmetric endomorphisms and symmetric matrices

An endomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is said *symmetric* or *self-adjoint* if

$$\langle f(\mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{u}, f(\mathbf{v})\rangle$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Under the standard correspondence between endomorphisms over $\mathbb{R}^n$ and matrices of $\mathbb{R}^{n\times n}$, symmetric transformations correspond to symmetric matrices. Note that the composition of two symmetric endomorphisms, as well as the product of two symmetric matrices, is generally not symmetric.

Among symmetric matrices, those being idempotent i.e. satisfying $P^2 = P$ are projection matrices. As we saw in the previous chapter, projection matrices of order n correspond to orthogonal projections onto subspaces of $\mathbb{R}^n$, which as a consequence, are linear transformations.

7.2.3 Scalar matrices and homotheties

A square matrix is called *scalar* if it is equal to λI_n, for a scalar $\lambda \in \mathbb{R}$. The corresponding endomorphism $f_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ is an homothety of ratio λ, if $\lambda \neq 0$ and the null transformation if $\lambda = 0$.

More generally, the endomorphism associated to a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ rescales the vector $\mathbf{e}_i$ of the standard basis by a factor d_i, for $i = 1, \ldots, n$.

Chapter 8

8 Eigenvalues and Eigenvectors

8.1 The characteristic polynomial of a matrix

A **non-null vector v** in $\mathbb{R}^n$ and a real number λ are, respectively, an *eigenvector* and an *eigenvalue* of a square $n \times n$ matrix A if

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{2}$$

Expression (2) can be rewritten as

$$A\mathbf{v} - \lambda I_n\mathbf{v} = \mathbf{0}$$

or

$$(A - \lambda I_n)\mathbf{v} = \mathbf{0} \tag{3}$$

with

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Expression (3) can be looked at as a linear homogeneous system with n equations and n unknowns, whose coefficient matrix $A - \lambda I_n$ depends on a parameter λ

$$(A - \lambda I_n)\mathbf{v} = \begin{bmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The eigenvector $\mathbf{v}$ is a non-trivial solution of the linear system (3) corresponding to a given value of λ. When λ changes, the vector of solutions changes. Every eigenvector $\mathbf{v}$ of a matrix $A \in \mathbb{R}^{n\times n}$ refers to only one eigenvalue, since if $A\mathbf{v} = \lambda_1\mathbf{v}$ and $A\mathbf{v} = \lambda_2\mathbf{v}$ with $\mathbf{v} \neq \mathbf{0}$, then $(\lambda_1 - \lambda_2)\mathbf{v} = \mathbf{0}$, hence $\lambda_1 - \lambda_2 = 0$, so $\lambda_1 = \lambda_2$.

In order for (3) to have a non-trivial solution, the rank of the coefficient matrix $A - \lambda I_n$ must be less than n, namely $A - \lambda I_n$ must be singular. We need to choose values of $\lambda \in \mathbb{R}$ such that

$$\det(A - \lambda I_n) = 0. \tag{4}$$

Expanding the determinant of (4) we get a polynomial of degree n with variable λ

$$p_A(\lambda) = \det(A - \lambda I_n) = (-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_2\lambda^2 + c_1\lambda + c_0$$

called the characteristic polynomial of A, and thus the eigenvalues of A are the roots of its characteristic polynomial, namely the solutions of the equation

$$(-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_2\lambda^2 + c_1\lambda + c_0 = 0.$$

We have $c_0 = \det A$ and $c_{n-1} = (-1)^{n-1}\mathrm{tr}(A)$, where $\mathrm{tr}(A)$ denotes the trace of A that is the sum $a_{11}+a_{22}+\cdots+a_{nn}$ of the elements on the main diagonal of A.

This equations will not always have solutions in $\mathbb{R}$. We are only interested in the real eigenvalues (if any) of a matrix A. The (eventually empty) set of all eingenvalues of a matrix A is called the spectrum of the matrix and is denoted with $\mathrm{spec}(A)$.

Example 8.1. 1. The characteristic polynomial of matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is

$$\det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)(4-\lambda)$$

and

$$(1-\lambda)(-1-\lambda)(4-\lambda) = 0 \Longleftrightarrow \lambda = 1, \lambda = -1, \lambda = 4.$$

The matrix A has three distinct real eigenvalues, $1, -1, 4$, that is $\mathrm{spec}(A) = \{1, -1, 4\}$.

2. The characteristic polynomial of $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

and the equation $\lambda^2 + 1 = 0$ has no solution in $\mathbb{R}$; matrix B has no real eigenvalues.

3. The characteristic polynomial of $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 8 & 6 & 4 & 0 \\ 1 & 2 & 0 & -3 \end{bmatrix}$ is

$$\det(C - \lambda I_4) = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ -2 & 1-\lambda & 0 & 0 \\ 8 & 6 & 4-\lambda & 0 \\ 1 & 2 & 0 & -3-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)(4-\lambda)(-3-\lambda).$$

The roots of the polynomial are $\lambda = 1, \lambda = 1, \lambda = 4, \lambda = -3$. Observe that $\lambda = 1$ appears twice among the solutions: $\lambda = 1$ is said to be *a root of multiplicity 2 (double)* of the characteristic polynomial. The eigenvalues of C are $\lambda = 1$ with *algebraic multiplicity* 2, and $\lambda = 4$, $\lambda = -3$, of algebraic multiplicity 1, i.e. *simple* eigenvalues.

4. The eigenvalues of matrix $H = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$ are the solutions of the equation $\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -\lambda & -2 \\ 0 & 2 & -\lambda \end{vmatrix} = (3-\lambda)(\lambda^2+4) = 0$. The only real eigenvalue of H is $\lambda = 3$.

We now turn our attention from the eigenvalues of A to its eigenvectors. Let λ^* be a real root of the characteristic polynomial; then the linear homogeneous system

$$(A - \lambda^* I_n)\mathbf{v} = \begin{bmatrix} a_{11}-\lambda^* & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda^* & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

has non-trivial solutions. The set of all solutions is a non-null subspace of $\mathbb{R}^n$ called the *eigenspace* of the eigenvalue λ^*

$$\mathbf{S}_{\lambda^*} = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \lambda^*\mathbf{v}\}.$$

The non-null vectors of $\mathbf{S}_{\lambda^*}$ are the eigenvectors of A belonging to eigenvalue λ^*. As is well-known

$$1 \leq \dim \mathbf{S}_{\lambda^*} = n - \operatorname{rank}(A - \lambda^* I_n).$$

We set $m_g(\lambda^*) = \dim \mathbf{S}_{\lambda^*}$ and call it the *geometric multiplicity* of λ^*, we have

$$\boxed{m_g(\lambda^*) = n - \operatorname{rank}(A - \lambda^* I_n).}$$

The adjective geometric is used to distinguish this multiplicity from the algebraic one, that is the multiplicity of λ^* as a root of the characteristic polynomial.

Example 8.2. 1. The matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ has spectrum $\{1, -1, 4\}$.

The eigenspace of the eigenvalue $\lambda_1 = 1$ is the subspace of $\mathbb{R}^n$ given by all the solutions to the linear homogeneous system

$$A - I_3 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

equivalent to

$$\begin{cases} y = 0 \\ z = 0 \end{cases}.$$

Thus

$$\mathbf{S}_1 = \{(x, 0, 0); x \in \mathbb{R}\},$$

and a basis of $\mathbf{S}_1$ is for instance $\mathbf{v}_1 = (1, 0, 0)$.

In a similar way one obtains the eigenspace of $\lambda_2 = -1$

$$\mathbf{S}_{-1} = \{(x, -x, 0); x \in \mathbb{R}\}$$

with relative basis $\mathbf{v}_2 = (1, -1, 0)$, and the eigenspace of $\lambda_3 = 4$

$$\mathbf{S}_4 = \{(x, 0, -3x); x \in \mathbb{R}\},$$

with basis $\mathbf{v}_3 = (1, 0, -3)$.

2. The eigenvalues of matrix $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 8 & 6 & 4 & 0 \\ 1 & 2 & 0 & -3 \end{bmatrix}$ are $\lambda_1 = 1$, with algebraic multiplicity 2, and $\lambda_2 = 4, \lambda_3 = -3$. The eigenspace of the eigenvalue $\lambda_1 = 1$ is the space of all solutions of the homogeneous system

$$\begin{cases} -2x = 0 \\ 8x + 6y + 3z = 0 \\ x + 2y - 4t = 0 \end{cases},$$

namely

$$\mathbf{S}_1 = \{(0, 2t, -4t, t); t \in \mathbb{R}\}$$

which is one-dimensional, that is $m_g(S_1) = 1$, with a basis given, for instance, by vector $\mathbf{v} = (0, 2, -4, 1)$. The eigenspace of $\lambda_2 = 4$ is

$$\mathbf{S}_4 = \{(0, 0, z, 0); z \in \mathbb{R}\},$$

with basis $\mathbf{u} = (0, 0, 1, 0)$. The eigenspace of $\lambda_3 = -3$ is

$$\mathbf{S}_{-3} = \{(0, 0, 0, t); t \in \mathbb{R}\},$$

with basis $\mathbf{w} = (0, 0, 0, 1)$.

The last example shows that the geometric and algebraic multiplicity of an eigenvalue can be different. Nevertheless, if we denote with $m_a(\lambda^*)$ the algebraic multiplicity of λ^* it is possible to prove that

$$\boxed{m_g(\lambda^*) \leq m_a(\lambda^*).}$$

8.2 Properties of eigenvalues and eigenspaces

We proceed to show some simple properties of eigenvalues and eigenvectors.

- The only eigenvalue of a scalar matrix λI_n is λ, with multiplicity (both algebraic and geometric) n. This means that $\mathbf{S}_\lambda = \mathbb{R}^n$, so each non-null vector is an eigenvector. Particularly I_n has only 1 as eigenvalue while the only eigenvalue of 0_n is 0.

- The eigenvalues of a diagonal matrix are the elements on the main diagonal, and the i-th standard vector $\mathbf{e}_i = (0, \dots, 0, 1, 0 \dots, 0)$ is an eigenvector relative to the i-th element of the main diagonal, for $i = 1, \dots, n$. The number of times that an eigenvalue appears on the main diagonal is equal to its multiplicity (both algebraic and geometric).

- The eigenvalues of an upper (or lower) triangular matrix are the elements on the main diagonal. The number of times that an eigenvalue appears on the main diagonal is equal to its **algebraic** multiplicity.

- A matrix A is singular $\iff$ the scalar $\lambda = 0$ is an eigenvalue of A since $\det A = \det(A - 0I_n) = 0$. Moreover the eigenspace of 0 is the nullspace of A, that is $\mathbf{S}_0 = \mathcal{N}(A)$.

- The matrices A and A^T have the same characteristic polynomial, hence the same eigenvalues

$$\det(A^T - \lambda I_n) = \det\left((A - \lambda I_n)^T\right) = \det(A - \lambda I_n).$$

- If A is non-singular, λ is an eigenvalue of $A \iff \lambda^{-1}$ is an eigenvalue of A^{-1}.
  Since A is non-singular, $\lambda \neq 0$, furthermore $A\mathbf{v} = \lambda\mathbf{v}$ implies $A^{-1}A\mathbf{v} = I_n\mathbf{v} = \lambda A^{-1}\mathbf{v}$ hence
  $$A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}.$$
  Note that $\mathbf{v}$ is an eigenvector both of A and of A^{-1}.

- If λ is an eigenvalue of $A \implies \lambda^p$ is an eigenvalue of A^p.
  From $A\mathbf{v} = \lambda\mathbf{v}$ there follows $A^2\mathbf{v} = \lambda A\mathbf{v} = \lambda^2\mathbf{v}$ and so on. The eigenspace of λ is contained in that of λ^p for A^p, for any $p > 0$.

- The eigenvalues of an orthogonal matrix, if any, are equal to ± 1.
  If $A\mathbf{v} = \lambda\mathbf{v}$, from $\|A\mathbf{v}\| = \|\mathbf{v}\|$ we get $\|\mathbf{v}\| = \|\lambda\mathbf{v}\| = |\lambda|\|\mathbf{v}\|$ and so $|\lambda| = 1$.

- The eigenvalues of an idempotent matrix H, if any, are equal to 0 and 1.
  If $H = H^2$ and $H\mathbf{v} = \lambda\mathbf{v}$ then also $H\mathbf{v} = \lambda\mathbf{v} = H^2\mathbf{v} = \lambda^2\mathbf{v}$, hence $(\lambda - \lambda^2)\mathbf{v} = \mathbf{0}$. Since $\mathbf{v} \neq \mathbf{0}$ then $\lambda - \lambda^2 = 0$ i.e. $\lambda = 1$ or $\lambda = 0$.
  Remember that a projection matrix is idempotent. In this case the eigenspace of 1 is the subspace of $\mathbb{R}^n$ onto which we are projecting, that is the column space of the matrix, while the eigenspace of 0 is its orthogonal complement, that is the left nullspace.

Remark Under the standard correspondence between matrices and endomorphism, if $\lambda \in \mathbb{R}$ is an eigenvalue of A then the restriction of f_A to the eigenspace $\mathbf{S}_\lambda$ is

- an homothety of ratio λ, if $\lambda \neq 0$
- the null transformation if $\lambda = 0$, that is $\mathbf{S}_0 = \ker f_A$.

In other word $\mathbf{v}$ is an eigenvector relative to a non zero eigenvalue if and only if $\mathbf{v}$ and $A\mathbf{v}$ are proportional (i.e. they share the same direction that is they belong to the same line); moreover the eigenvalue is the proportional coefficient. With this in mind, a non-null vector $\mathbf{v}$ in $\mathbb{R}^n$ and a real number λ are, respectively, an *eigenvector* and an *eigenvalue* of an endomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ if $f(\mathbf{v}) = \lambda\mathbf{v}$, that is if they are, respectively, an eigenvector and an eigenvalue of the matrix A that (canonically) represents f, i.e. such that $f_A = f$.

8.3 Diagonalizing a matrix

We already saw that every eigenvector refers to only one eigenvalue, moreover we have the following result.

Proposition 8.1. *Eigenvectors of a matrix A corresponding to distinct eigenvalues are linearly independent.*

The maximum number of linearly independent eigenvectors for a matrix A of order n is clearly n: when this is the case, we will prove that the following equality holds

$$D = E^{-1}AE$$

where E is a regular matrix having the eigenvectors as columns and D is a diagonal matrix having the eigenvalues on the main diagonal. Before stating the result, we introduce a new relation between matrices, motivated by the above equation. If $A, B \in \mathbb{R}^{n\times n}$ are two square matrices B is said to be *similar* to A if there exists a non-singular matrix E such that

$$B = E^{-1}AE.$$

This is an equivalence relation[4] on the set of matrices of order n.

Proposition 8.2. *Similar matrices have the same characteristic polynomial, and hence the same eigenvalues with the same algebraic multiplicities.*

Proof If $B = E^{-1}AE$:

$$\det(B - \lambda I_n) = \det(E^{-1}AE - \lambda I_n) = \det(E^{-1}AE - \lambda E^{-1}I_nE) =$$

$$= \det\left(E^{-1}(A - \lambda I_n)E\right) = (\det E)^{-1}\det(A-\lambda I_n)(\det E) = \det(A-\lambda I_n). \qquad \blacksquare$$

[4] A relation over a set is an equivalence relation if it is reflexive, symmetric and transitive.

As a consequence similar matrices have also the same determinant and the same trace. On the contrary eigenspaces relative to the same eigenvalue are in generally different: nevertheless they have the same dimension, that is the geometric multiplicity. Hence similar matrices share also the rank.

Going back to the above problem, for any real square matrix A, we wonder whether A can be *to put in a diagonal form* namely if A is similar to a diagonal matrix D, which means that there exists a non-singular real matrix E such that

$$E^{-1}AE = D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_n \end{bmatrix}.$$

If A is similar to a diagonal matrix D, calculations are simplified

- the eigenvalues of A are the same as those of D, which are the elements along the main diagonal of D;
- since rank$A$ =rank$D$, the rank of A is equal to the number of non-zero elements along the diagonal of D, hence to the number of non-zero eigenvalues of A;
- $\det A = \det D = d_1 d_2 \cdots d_n$;
- if $E^{-1}AE = D$ then also $A = EDE^{-1}$ so that

  $$A^2 = (EDE^{-1})^2 = (EDE^{-1})(EDE^{-1}) = EDI_nDE^{-1} = ED^2E^{-1},$$

  and

  $$A^k = (EDE^{-1})^k = (EDE^{-1})(EDE^{-1})\dots(EDE^{-1}) = ED^kE^{-1}.$$

I f a matrix is similar to a diagonal matrix we will say that it is *diagonalizable.* Now we can state the result announced at the beginning of the section.

Theorem 8.1 (Fundamental Theorem of diagonalization)**.** *An $n \times n$ matrix A is diagonalizable if and only if there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A (i.e. there exists n linearly independent eigenvectors).*

Proof If A can be diagonalized and D is a diagonal matrix similar to A, namely

$$E^{-1}AE = D,$$

with $E = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix}$, let us find out the meaning of the columns of matrix E. Since E is non-singular, its columns are linearly independent vectors of $\mathbb{R}^n$. Clearly

$$AE = A\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{c}_1 & A\mathbf{c}_2 & \cdots & A\mathbf{c}_n \end{bmatrix} =$$

$$= ED = \begin{bmatrix} d_1\mathbf{c}_1 & d_2\mathbf{c}_2 & \cdots & d_n\mathbf{c}_n \end{bmatrix},$$

hence

$$A\mathbf{c}_i = d_i\mathbf{c}_i$$

for $i = 1, \ldots, n$. The conclusion is that if A can be diagonalized, then the columns of E are the eigenvectors corresponding to the elements on the diagonal of D, which are the eigenvalues of A. Thus A has n independent eigenvectors.

To show the converse, assume that

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

is a basis of $\mathbb{R}^n$, and the vectors $\mathbf{v}_i$ are eigenvectors of A

$$A\mathbf{v}_i = \lambda_i\mathbf{v}_i.$$

Let

$$E = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

and

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Then

$$AE = A\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{bmatrix};$$

$$ED = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{bmatrix}$$

hence

$$AE = ED,$$

and so

$$E^{-1}AE = D. \qquad \blacksquare.$$

It follows from the above that if a matrix A can be put in diagonal form $E^{-1}AE = D$, then

- the sum of the eigenvalues of A is the trace of A,
- the product of the eigenvalues of A is the determinant of A.

It is possible to prove that these properties hold in general, even if it is not possible to put A in a diagonal form.

Example 8.3. 1. Matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ can be diagonalized; if we choose a basis in each of the three eigenspaces of A we find

$$\{\mathbf{v}_1 = (1,0,0), \mathbf{v}_2 = (1,-1,0), \mathbf{v}_3 = (1,0,-3)\}$$

made up of eigenvectors of A belonging, respectively, to $\lambda = 1, \lambda = -1, \lambda = 4$. They are linearly independent vectors because they are eigenvectors of A belonging to different eigenvalues; so this is a basis of $\mathbb{R}^3$ consisting in eigenvectors of A.

A non singular matrix which diagonalizes A is, for instance, $E = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, the diagonal matrix similar to A is $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

2. Matrix $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ cannot be diagonalized over $\mathbb{R}$, since it has no real eigenvalues, nor eigenvectors.

3. Matrix $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 8 & 6 & 4 & 0 \\ 1 & 2 & 0 & -3 \end{bmatrix}$ cannot be diagonalized; there is only one independent eigenvector belonging to the eigenvalue $\lambda = 1$, namely the eigenspace S_1 has dimension 1; similarly for $\lambda = 4$ e $\lambda = -3$. Thus the maximum number of linearly independent eigenvectors of A is three and they cannot form a basis of $\mathbb{R}^4$.

4. The matrix $H = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$ cannot be diagonalized. The only real eigenvalue of H is $\lambda = 3$. The relative eigenspace $\mathbf{S}_3 = \{(x,0,0) \mid x \in \mathbb{R}\}$ has dimension 1, it is impossible to find a basis of $\mathbb{R}^3$ formed by eigenvectors of H.

5. The characteristic polynomial of the matrix

$$M = \begin{bmatrix} 9 & 0 & 10 \\ -5 & 4 & -10 \\ -5 & 0 & -6 \end{bmatrix}$$

is

$$\det \begin{bmatrix} 9-\lambda & 0 & 10 \\ -5 & 4-\lambda & -10 \\ -5 & 0 & -6-\lambda \end{bmatrix} = (4-\lambda)(\lambda^2 - 3\lambda - 4) = -(4-\lambda)^2(\lambda+1).$$

The eigenvalues of M are $\lambda = 4$ with algebraic multiplicity 2 and $\lambda = -1$ with algebraic multiplicity 1 (simple). The eigenspace of M belonging to $\lambda = 4$ is the subspace of $\mathbb{R}^4$ given by the solutions of the equation

$$5x + 10z = 0,$$

hence

$$\mathbf{S}_4 = \{(-2\alpha, \beta, \alpha) \mid \alpha, \beta \in \mathbb{R}\}.$$

It has dimension 2 and a possible basis of $\mathbf{S}_4$ is for instance

$$\{\mathbf{v}_1 = (-2, 0, 1), \mathbf{v}_2 = (0, 1, 0)\}.$$

The eigenspace belonging to $\lambda = -1$ is:

$$\mathbf{S}_{-1} = \{(-\alpha, \alpha, \alpha) \mid \alpha \in \mathbb{R}\},$$

and a basis is

$$\{\mathbf{v}_3 = (-1, 1, 1)\}.$$

The set obtained putting these bases together

$$\{\mathbf{v}_1 = (-2, 0, 1), \mathbf{v}_2 = (0, 1, 0), \mathbf{v}_3 = (-1, 1, 1)\}$$

is a basis of $\mathbb{R}^3$ consisting in eigenvectors of M. A matrix that diagonalizes M is

$$E = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

The diagonal matrix similar to M is

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

6. The matrix

$$N = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

is not diagonalizable. The only eigenvalue is $\lambda = 3$ with algebraic multiplicity 2; the correspondent eigenspace is $\mathbf{S}_3 = \{(0, \alpha) \mid \alpha \in \mathbb{R}\}$ and has dimension 1; it is not possible to find two linearly independent eigenvectors.

If A is diagonalizable, to construct D, E such that $D = E^{-1}AE$, the order of the eigenvalues on the main diagonal of D must correspond to the order of the relative eigenvectors as columns of the matrix E that diagonalizes A.

Recalling that eigenvectors relative to different eigenvalues are linearly independent, a sufficient but not necessary condition to diagonalize a matrix is the following.

Proposition 8.3. *A square matrix of order n having n (distinct) eigenvalues is diagonalizable*

More generally, it is easy to prove (using the fundamental theorem of diagonalization), that a matrix of order n **on the real number** is diagonalizable if and only if

(a) all the roots of the characteristic polynomial of A are real

$$\det(A - \lambda I_n) = (-1)^n(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \dots (\lambda - \lambda_r)^{m_r}$$

where $\text{spec}(A) = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$ and $m_i = m_a(\lambda_i)$ is the algebraic multiplicity of λ_i, with $i = 1, \dots, r$, that is

$$m_1 + m_2 + \cdots + m_r = n.$$

(b) The eigenspace $\mathbf{S}_{\lambda_i}$ has dimension m_i, that is to say, the geometric multiplicity of λ_i is equal to the algebraic one, for $i = 1, \dots, r$.

Moreover, note that properties (a) and (b) together are equivalent to the following one

$$\boxed{m_g(\lambda_1) + \cdots + m_g(\lambda_r) = n.} \tag{5}$$

If A is diagonalizable, the matrices E and D such that $D = E^{-1}AE$ are sometimes called *eigenvector* matrix and *eigenvalue* matrix, respectively. Notice that they are not unique: by permuting the columns of E and the elements of the main diagonal of D in the same way, we obtain a different couple of eigenvector-eigenvalue matrices.

Remark What does the equality $B = E^{-1}AE$ means in terms of endomorphisms? How are connected the endomorphisms f_A and f_B, when A and B are similar? We have $f_B = f_{E^{-1}} f_A f_E$, so, to answer, we have to guess the meaning of f_E. Since E is regular, the columns $\mathcal{C} = (\mathbf{c}_1, \dots, \mathbf{c}_n)$ of E are an ordered basis of $\mathbb{R}^n$ and, under the canonical correspondence $E \longleftrightarrow f_E$, are the image of the standard basis. So if $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ is any vector, then

$$\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n \Longrightarrow f_E(\mathbf{v}) = v_1 f_E(\mathbf{e}_1) + \cdots v_n + f_E(\mathbf{e}_n) = v_1\mathbf{c}_1 + \cdots + v_n\mathbf{c}_n.$$

so a f_E sends a vector having coordinates[5] $(v_1, \dots, v_n)$ with respect to the standard basis into a vector having the same coordinates with respect to the basis $\mathcal{C}$: we can interpret f_E as a "change of basis" from the canonical basis to $\mathcal{C}$. Hence $f_{E^{-1}}$ will be a change of basis in the opposite direction (i.e. from $\mathcal{C}$ to the canonical one). So the interpretation of the equality $B = E^{-1}AE$ in terms of endomorphisms is that, **up to a change of basis**, f_A operates as f_B.

[5]Given an ordered basis $\mathcal{B} = (\mathbf{w}_1, \dots \mathbf{v}_n)$, any vector $\mathbf{v}$ can be expressed as a linear combination $\mathbf{v} = a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n$ of the vectors of the basis in a unique way. We say that the ordered weights $(a_1, \dots, a_n)$ are the *coordinates* of $\mathbf{v}$ with respect to $\mathcal{B}$.

8.4 Diagonalizing symmetric matrices or self-adjoint endomorphisms

If P is a projection matrix, equality (5) holds, since P has only two eigenvalues $1, 0$ whose eigenspaces are, respectively, the subspace onto which P projects (i.e. $\mathcal{C}(P)$) and its orthogonal complement (i.e. $\mathcal{N}(P^T)$). As a consequence each projection matrix is diagonalizable. More generally, all symmetric matrices are diagonalizable, as proved by the following results.

Proposition 8.4. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then*

1. *all the roots of the characteristic polynomial, i.e. all the eigenvalues, are real;*
2. *every pair of eigenvectors of A belonging to distinct eigenvalues are orthogonal.*

Proof We will prove only the second statement. Let λ_1, λ_2 be two distinct eigenvalues of A and $\mathbf{v}_1, \mathbf{v}_2$ two eigenvectors of A belonging, respectively, to λ_1 and λ_2

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \quad A\mathbf{v}_2 = \lambda_2\mathbf{v}_2.$$

We have

$$\lambda_1\langle\mathbf{v}_1, \mathbf{v}_2\rangle = \langle\lambda_1\mathbf{v}_1, \mathbf{v}_2\rangle = \langle A\mathbf{v}_1, \mathbf{v}_2\rangle = \langle\mathbf{v}_1, A\mathbf{v}_2\rangle = \langle\mathbf{v}_1, \lambda_2\mathbf{v}_2\rangle = \lambda_2\langle\mathbf{v}_1, \mathbf{v}_2\rangle$$

Since the eigenvalues λ_1 and λ_2 are different there follows $\langle\mathbf{v}_1, \mathbf{v}_2\rangle = 0$. The eigenspaces S_{λ_1} and $S_{\lambda_2}$are thus orthogonal; every vector of the first one is orthogonal to every vector of the second one. ■

A fundamental result of linear algebra is the following.

Theorem 8.2 (Spectral Theorem)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix then there exists an orthonormal basis of $\mathbb{R}^n$ made up of eigenvectors of A. An equivalent statement is: if A is symmetric, there exist an orthogonal matrix Q and a diagonal matrix Λ such that*

$$\boxed{Q^T AQ = Q^{-1}AQ = \Lambda.}$$

So any symmetric matrix is diagonalizable and it is possible to find an orthonormal basis of eigenvectors. In terms of endomorphisms this means that, **up to linear isometries**, each self-adjoint endomorphism behaves as a diagonal endomorphism.

Example 8.4. 1. The symmetric matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

has, as expected, three real eigenvalues: $\lambda = 2, \lambda = 0, \lambda = -2$ each with multiplicity 1. The relative eigenspaces are

$$\begin{aligned}\mathbf{S}_2 &= \{y(1,1,0); y \in \mathbb{R}\},\\ \mathbf{S}_0 &= \{y(-1,1,0); y \in \mathbb{R}\},\\ \mathbf{S}_{-2} &= \{z(0,0,1); z \in \mathbb{R}\},\end{aligned}$$

and the relative bases are

$$\{\mathbf{v}_1 = (1,1,0)\}, \{\mathbf{v}_2 = (-1,1,0)\}, \{\mathbf{v}_3 = (0,0,1)\}.$$

To find an orthonormal basis of each of the eigenspaces, we need to normalize the vectors of each basis

$$\mathbf{v}_1^* = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), \mathbf{v}_2^* = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), \mathbf{v}_3^* = (0,0,1).$$

The set of orthonormal vectors

$$\left\{\mathbf{v}_1^* = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), \mathbf{v}_2^* = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), \mathbf{v}_3^* = (0,0,1)\right\}$$

is an orthonormal basis of the Euclidean space $\mathbb{R}^3$. An orthogonal matrix that makes A diagonal is

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

whereas the diagonal matrix Λ similar to A is:

$$\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

2. The eigenvalues of the symmetric matrix

$$B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

are $\lambda = 6$ (with multiplicity 2) and $\lambda = 1$ (simple). The homogeneous system that gives the eigenspace relative to $\lambda = 6$ consists of just one equation

$$2y + z = 0,$$

thus

$$\mathbf{S}_6 = \{(x, y, -2y) \quad x, y \in \mathbb{R}\}.$$

A basis of $\mathbf{S}_6$ is

$$\{\mathbf{v}_1 = (1,0,0), \mathbf{v}_2 = (0,1,-2)\};$$

which is already orthogonal (if not, we can use the Gram-Schmidt process to obtain an orthogonal basis); an orthonormal basis of S_6 is obtained normalizing the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$

$$\{\mathbf{v}_1^* = (1,0,0), \mathbf{v}_2^* = (0, \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})\}.$$

The eigenspace belonging to the eigenvalue $\lambda = 1$ is

$$\mathbf{S}_1 = \{(0, 2z, z), z \in \mathbb{R}\},$$

an orthonormal basis is

$$\{\mathbf{v}_3^* = (0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})\}.$$

The set

$$\{\mathbf{v}_1^* = (1,0,0), \mathbf{v}_2^* = (0, \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}), \mathbf{v}_3^* = (0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})\}$$

is an orthonormal basis of $\mathbb{R}^3$ made of eigenvectors of B. A spectral decomposition of B is

$$B = Q\Lambda Q^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}.$$

Rephrasing the Spectral theorem, each symmetric matrix A admits a decomposition as

$$A = Q\Lambda Q^T,$$

where Q is an orthogonal matrix and Λ is a diagonal matrix. Such a decomposition, which is not unique, is called *spectral decomposition* and characterizes symmetric matrices: any square matrix admitting such a decomposition is symmetric. Clearly Λ and Q are not unique: a convention sometimes used is to order the eingevalues, along the main diagonal of Λ, from the greatest to the lowest.

Let $A \in \mathcal{S}_n(\mathbb{R})$ having n different eigenvalues and consider a spectral decomposition of $Q\Lambda Q^T$ of A with $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and $Q = [\mathbf{q}_1 \ \mathbf{q}_2 \cdots \mathbf{q}_n]$. For any $\mathbf{v} \in \mathbb{R}^n$, we have

$$\begin{aligned} A\mathbf{v} &= Q\Lambda Q^T\mathbf{v} = [\lambda_1\mathbf{q}_1 \ \lambda_2\mathbf{q}_1 \cdots \lambda_n\mathbf{q}_n] \begin{bmatrix} \mathbf{q}_1^T\mathbf{v} \\ \mathbf{q}_2^T\mathbf{v} \\ \vdots \\ \mathbf{q}_n^T\mathbf{v} \end{bmatrix} \\ &= (\lambda_1\mathbf{q}_1)\mathbf{q}_1^T\mathbf{v} + (\lambda_2\mathbf{q}_2)\mathbf{q}_2^T\mathbf{v} + \cdots + (\lambda_n\mathbf{q}_n)\mathbf{q}_n^T\mathbf{v} = (\lambda_1\mathbf{q}_1\mathbf{q}_1^T + \lambda_2\mathbf{q}_2\mathbf{q}_2^T + \cdots + \lambda_n\mathbf{q}_n\mathbf{q}_n^T)\mathbf{v} \end{aligned}$$

so

$$A = \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \lambda_2 \mathbf{q}_2 \mathbf{q}_2^T + \cdots + \lambda_n \mathbf{q}_n \mathbf{q}_n^T.$$

Since $\mathbf{q}_i$ is an orthonormal basis of $\mathbf{S}_{\lambda_i}$, the matrix $\mathbf{q}_i \mathbf{q}_i^T$ is the projection matrix onto $\mathbf{S}_{\lambda_i}$. So, A can be expressed as a linear combination of the n (symmetric and idempotent) projection matrices onto the eigenspaces with weights the eigenvalues. The same decomposition holds in the general case, that is if A is a symmetric matrix then it decomposes as

$$\boxed{A = \lambda_1 P_{\mathbf{S}_{\lambda_1}} + \lambda_2 P_{\mathbf{S}_{\lambda_2}} + \cdots \lambda_k P_{\mathbf{S}_{\lambda_k}}}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the eigenvalues of A and $P_{\mathbf{S}_{\lambda_i}}$ is the projection matrix onto the eigenspace S_{λ_i}, for $i = 1, \ldots, k$. Also this additive decomposition is referred to as *spectral decomposition* of A. Notice that in this additive form the spectral decomposition is unique.

Example 8.5. The eigenvalues of the symmetric matrix

$$B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

are, as we have already computed, $\lambda = 6$ (with multiplicity 2) and $\lambda = 1$ (simple). Then

$$B = 6P_6 + 1P_1,$$

where P_6, P_1 are the projection matrices onto the eigenspaces $\mathbf{S}_6, \mathbf{S}_1$. Hence

$$P_6 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{5}} \\ 0 & -\frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} \qquad P_1 = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

and so

$$B = 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & -\frac{2}{5} \\ 0 & -\frac{2}{5} & \frac{4}{5} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{2}{5} \\ 0 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}.$$

8.5 Real quadratic forms

A *real quadratic form* over $\mathbb{R}^n$ is a function $q : \mathbb{R}^n \to \mathbb{R}$ such that

$$q(x_1, \ldots, x_n) = (x_1 \ \ldots \ x_n) A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

where A is a real symmetric matrix of order n called *matrix of the quadratic form*. As a consequence, any quadratic form can be expressed as

$$q(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_{ii} x_i^2 + 2 \sum_{i \neq j = 1}^{n} a_{ij} x_i x_j,$$

that is, as an homogeneous polynomial of degree 2 in the unknowns $x_1, \ldots, x_n$.

Example 8.6.

1. The all the quadratic forms over, respectively, $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ have the following form

$$q(x) = ax^2,$$
$$q(x, y) = ax^2 + 2bxy + cy^2,$$
$$q(x_1, x_2, x_3) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + a_{33}x_3^2.$$

2. The symmetric matrix
$$S = \begin{bmatrix} 4 & 0 & 3 \\ 0 & 6 & 1 \\ 3 & 1 & -1 \end{bmatrix}$$
defines the quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$
$$q(x_1, x_2, x_3) = \mathbf{x}^T S \mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} S \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4x_1^2 + 6x_2^2 + 6x_1x_3 + 2x_2x_3 - x_3^2.$$

3. The matrix of the quadratic form $q : \mathbb{R}^2 \to \mathbb{R}$ defined by $q(x, y) = 5x^2 - 3xy + y^2$ is
$$\begin{bmatrix} 5 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{bmatrix}.$$

A quadratic form $q : \mathbb{R}^n \to \mathbb{R}$ has the following properties

1. $q(\mathbf{0}) = 0$,
2. $q(\alpha\mathbf{x}) = \alpha^2 q(\mathbf{x})$, with $\alpha \in \mathbb{R}$
3. if $q(\mathbf{v}) = 0$ then $q(\alpha\mathbf{v}) = 0$ for all $\alpha \in \mathbb{R}$; if $q(\mathbf{v}) \neq 0$ then $q(\mathbf{v})$ and $q(\alpha\mathbf{v})$ have the same sign.

The last property leads to a classification of quadratic forms depending on their *sign*. We will say that the quadratic form $q : \mathbb{R}^n \to \mathbb{R}$, as well as its matrix, is

1. *positive definite* if $q(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n$, with $\mathbf{x} \neq \mathbf{0}$
2. *positive semidefinite* if $q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$
3. *negative definite* if $q(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathbb{R}^n$, with $\mathbf{x} \neq \mathbf{0}$
4. *negative semidefinite* if $q(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \mathbb{R}^n$
5. *indefinite* if q assumes both negative and positive values.

If we consider a quadratic form $q : \mathbb{R}^n \to \mathbb{R}$

$$q(x) = a_{11}x_1^2 + a_{12}x_2^2 + \cdots + a_{nn}x_n^2$$

having a no cross-product term (i.e. the coefficients of the x_ix_j terms are zero), finding its sign is very easy

1. if $a_{ii} > 0$, for each $i = 1, \ldots, n$, the quadratic form is positive definite
2. if $a_{ii} \geq 0$, for each $i = 1, \ldots, n$, the quadratic form is positive semidefinite
3. if $a_{ii} < 0$, for each $i = 1, \ldots, n$, the quadratic form is negative definite
4. if $a_{ii} \leq 0$, for each $i = 1, \ldots, n$, the quadratic form is negative semidefinite
5. if at least one coefficient is positive and another one is negative the quadratic form is indefinite.

Example 8.7. The quadratic form

$$q(x_1, x_2, x_3, x_4) = 2x_1^2 + x_2^2 + 5x_3^2 + 7x_4^2$$

is positive definite, as well as the well known quadratic form $q(x_1, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2$ given by the square of the norm of $(x_1, x_2, \cdots, x_n)$.
The quadratic form

$$q(x_1, x_2, x_3, x_4) = 2x_1^2 + x_2^2 + 5x_4^2$$

is positive semidefinite, but not positive definite: in fact $q(0, 0, 1, 0) = 0$.
The quadratic form

$$q(x_1, x_2, x_3, x_4) = 2x_1^2 - x_2^2 + 5x_3^2 - x_4^2$$

is indefinite: $q(0, 1, 0, 0) = -1 < 0$ while $q(1, 0, 0, 0) = 2 > 0$.

Notice that the entries of the main diagonal of a symmetric matrix are the values of the quadratic form on the vectors of the standard basis. So for example

$$q(\mathbf{x}) = \mathbf{x}^T \begin{pmatrix} -1 & 3 & 4 \\ 3 & 4 & 0 \\ 4 & 0 & -5 \end{pmatrix} \mathbf{x}$$

is indefinite since $q(1, 0, 0) = -1 < 0$ and $q(0, 1, 0) = 4 > 0$. On the contrary we cannot say that the matrix

$$A = \begin{pmatrix} -1 & 3 & 4 \\ 3 & -4 & 0 \\ 4 & 0 & -5 \end{pmatrix}$$

is negative definite just looking at its main diagonal, because the signs of a quadratic form on vectors of the standard basis, does not determine the signs on the other vectors. In this case, for example, $q(1, 1, 1) = 4 > 0$, so also this matrix is indefinite.

Let $A = Q\Lambda Q^T$ be a spectral decomposition of A then

$$q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = \mathbf{x}^T Q \Lambda Q^T \mathbf{x} = (Q^T \mathbf{x})^T \Lambda (Q^T \mathbf{x}).$$

Operating the (orthogonal) *change of variable*

$$Q^T\mathbf{x} = Q^{-1}\mathbf{x} = \mathbf{y} \Longleftrightarrow \mathbf{x} = Q\mathbf{y}$$

the quadratic form becomes

$$q(\mathbf{x}) = \mathbf{y}^T\Lambda\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

Hence, by replacing the x_i's unknowns with the y_i's unknowns we obtain an easier form for q having no cross-product terms, , called *canonical metric form.* Notice that the coefficients are exactly the eigenvalues of A. More formally, we have the following result.

Theorem 8.3 (Principal Axes Theorem)**.** *Let A be a real symmetric $n \times n$ matrix. Then there exists an orthogonal change of variable $\mathbf{x} = Q\mathbf{y}$, that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into a quadratic form $\mathbf{y}^T\Lambda y$ with no cross-product terms.*

The columns of Q in the theorem are called *principal axes* of the quadratic form and are an orthonormal spectral basis of $\mathbb{R}^n$. The vector $\mathbf{y}$ contains the coefficients giving $\mathbf{x}$ as linear combination of the principal axes. The Principal Axes Theorem gives us a way to find the sign of a quadratic form.

Theorem 8.4. *The quadratic form $q : \mathbb{R}^n \to \mathbb{R}$ defined by $q(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x}$ and its matrix A are*

1. *positive definite if and only if all the eigenvalues of A are positive*
2. *positive semidefinite if and only if all the eigenvalues of A are not negative*
3. *negative definite if and only if all the eigenvalues of A are negative*
4. *negative semidefinite if and only if all the eigenvalues of A are not positive*
5. *indefinite if and only if A has both positive and negative eigenvalues.*

Proof By the Principal Axes Theorem there is an orthogonal change of variable $\mathbf{x} = Q\mathbf{y}$ such that

$$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T\Lambda\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Since Q is invertible there is bijection between all nonzero $\mathbf{x}$ and all nonzero $\mathbf{y}$. Thus the values of q over $\mathbf{x}$ correspond to the values of $\mathbf{y}^T\Lambda\mathbf{y}$ over $\mathbf{y}$, which are clearly controlled by the signs of the eigenvalues $\lambda_1, \ldots, \lambda_n$ as stated in the theorem. ■

Example 8.8. 1. The quadratic form $q(x_1, x_2) = x_1^2 - 2x_1x_2 + x_2^2$ is positive semidefinite and not definite. In fact

$$q(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and A has $\lambda = 2$, $\lambda = 0$ as eigenvalues, which are both non negative. A spectral decomposition of A is

$$A = Q\Lambda Q^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The principal axis are $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and the change of variable

$$Q^T\mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \mathbf{x} = \mathbf{y}$$

gives the metric canonical form

$$\mathbf{y}^T\Lambda\mathbf{y} = 2y_1^2.$$

2. The quadratic form $q(x_1, x_2) = x_1^2 + x_2^2 + 10x_1x_2$ is indefinite, since the eigenvalues of its matrix $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$ are one positive $\lambda = 6$, and one negative $\lambda = -4$.

We say that a symmetric matrix A has *signature* $(\sigma_0, \sigma_+, \sigma_-)$ if the multiplicity of the zero eigenvalue is σ_0 and it has σ_+ (resp. σ_-) positive (negative) eigenvalues, counted with multiplicity. The natural numbers $\sigma_0, \sigma_+, \sigma_-$, respectively, are also called *null index, positive index and negative index* of A. A quadratic form is completely determined by its signature, up to orthogonal change of basis and rescaling. For example, $\sigma_0 + \sigma_+ + \sigma_-$ is equal to the order of the matrix and $\sigma_+ + \sigma_-$ equals the rank. Moreover, using an algebraic result called Descartes' Theorem, it is possible to prove that σ_+ of A is equal to the number of *change of signs*[6] of the characteristic polynomial of A. Using this result it is possible to find the signature without computing the roots of the characteristic polynomial (i.e. the eigenvalues) which, in general, could be difficult or even not possible (if the degree is greater than 4).

In calculus, when dealing with multivariable smooth functions $f : \mathbb{R}^n \to \mathbb{R}$, we can associate to each $\mathbf{x} \in \mathbb{R}^n$ a symmetric matrix, called Hessian matrix: maximum points and minimum points correspond, respectively, to critical points whose associated matrix is negative or positive definite. The last result gives a different criterion to understand whether a symmetric matrix is positive or negative definite.

Theorem 8.5 (Sylvester's Criterion)**.** *Let A be a real symmetric matrix of order n. Denote with M_k the $k \times k$ minor of A identified by the first k rows and k columns (also called principal minor of order k), for $i = 1, \ldots, n$. The matrix A is*

[6] If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ is a real polynomial, a *change of sign* is a couple of non zero consecutive coefficients having different signs.

- *positive definite if and only if* $\det M_i$ *is a positive number for all* $i = 1, \dots, n$*;*
- *negative definite if and only if* $\det M_i$ *is a positive number for all the even* i *and a negative number for all the odd* i*, for* $i = 1, \dots, n$*.*

Since $M_n = A$, a positive definite matrix has always positive determinant, while a negative definite matrix of order n has negative determinant only if n is odd (and has positive determinant otherwise).

Example 8.9. The matrix

$$A = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

is positive definite, since

$$\det M_1 = a_{11} = 5 > 0 \quad \det M_2 = \det \begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix} = 14 > 0 \quad \det M_3 = \det A = 9 > 0.$$

On the contrary, the matrix

$$B = \begin{pmatrix} -5 & 1 & 0 \\ 1 & -3 & -1 \\ 0 & -1 & -1 \end{pmatrix}$$

is negative definite, since

$$\det M_1 = b_{11} = -5 < 0 \quad \det M_2 = \det \begin{pmatrix} -5 & 1 \\ 1 & -3 \end{pmatrix} = 14 > 0 \quad \det M_3 = \det B = -9 < 0.$$

Finally, the matrix

$$C = \begin{pmatrix} -5 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & -1 \end{pmatrix}$$

is neither negative nor positive definite, since

$$\det M_1 = c_{11} = -5 < 0 \quad \det M_2 = \det \begin{pmatrix} -5 & 1 \\ 1 & 3 \end{pmatrix} = -16 < 0.$$